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LETTER TO THE EDITOR

Significance of $c/\sqrt{2}$ in relativistic physics

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Abstract
In the description of *relative* motion in accelerated systems and gravitational fields, inertial and tidal accelerations must be taken into account, respectively. These involve a critical speed that in the first approximation can be simply illustrated in the case of motion in one dimension. For one-dimensional motion, such first-order accelerations are multiplied by $\left(1-V^2/V_c^2\right)$, where $V_c = c/\sqrt{2}$ is the critical speed. If the speed of relative motion exceeds V_c, there is a sign reversal with consequences that are contrary to Newtonian expectations.

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In four recent papers on the generalized Jacobi equation [1–4], we have considered the consequences of general relativity for the *relative* motion of nearby timelike geodesics when the speed of relative motion is arbitrary but of course less than c. Our main results, which can be clearly seen in the case of one-dimensional motion, depend on whether the speed of relative motion is above or below the critical speed $V_c = c/\sqrt{2} \approx 0.7c$. For one-dimensional motion with relative velocity V, the tidal force acting on a particle is multiplied by the factor $(1 - 2V^2/c^2)$, thus leading to tidal effects for $|V| > c/\sqrt{2}$ that are counterintuitive when compared to Newtonian expectations. Specifically, starting from the generalized Jacobi equation (which includes only first-order tidal effects) in a Fermi coordinate system $(T, \boldsymbol{X})$ established along the reference geodesic and restricting attention to a two-dimensional world (T, Z), the equation of relative motion reduces to

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}T^2} + \kappa\left(1 - 2\frac{V^2}{c^2}\right) Z = 0, \tag{1}$$

where $\kappa(T) = {}^F R_{TZTZ}$ is the Gaussian curvature of the surface (T, Z) evaluated along the worldline of the reference geodesic and $V = \mathrm{d}Z/\mathrm{d}T$ (see [1]).

The purpose of this letter is to demonstrate that the critical speed $V_c = c/\sqrt{2}$ appears also in the physics of translationally accelerated systems in Minkowski spacetime;

that is, it is a general feature of the theory of relativity. To this end, we consider two nearby worldlines in Minkowski spacetime: a reference worldline $\mathcal{O}$ that is arbitrarily accelerated and a geodesic worldline $\mathcal{A}$; we are interested in the motion of the free particle $\mathcal{A}$ relative to the noninertial observer $\mathcal{O}$. Referring to the observer's accelerated system, the motion of $\mathcal{A}$ is subject to translational and rotational inertial accelerations. We will show that relativistic inertial accelerations exist that go beyond Newtonian mechanics and are essentially due to the translational acceleration of $\mathcal{O}$. In this case, the inertial acceleration of $\mathcal{A}$ in the first approximation is multiplied by $(1 - 2V^2/c^2)$ for motion along the direction of translational acceleration.

Imagine an arbitrary accelerated observer $\mathcal{O}$ following a worldline $\bar{x}^\mu(\tau)$ in Minkowski spacetime. Here $x^\mu = (t, x, y, z)$ are inertial coordinates in the background global frame and τ is the proper time of the observer, i.e. $-\mathrm{d}\tau^2 = \eta_{\alpha\beta}\,\mathrm{d}\bar{x}^\alpha\,\mathrm{d}\bar{x}^\beta$, where $\eta_{\alpha\beta}$ is the Minkowski metric tensor with signature +2. Unless otherwise specified, we choose units such that $c = 1$. The observer has four-velocity $u^\mu = \mathrm{d}\bar{x}^\mu/\mathrm{d}\tau$ and translational acceleration $A^\mu(\tau) = \mathrm{d}u^\mu/\mathrm{d}\tau$. Moreover, at each instant of time along its worldline the observer is endowed with an orthonormal tetrad frame $\lambda^\mu{}_{(\alpha)}$ such that $\lambda^\mu{}_{(0)} = u^\mu$ and

$$\eta_{\mu\nu}\lambda^\mu{}_{(\alpha)}\lambda^\nu{}_{(\beta)} = \eta_{(\alpha)(\beta)}. \tag{2}$$

The variation of this tetrad along the worldline of the observer is given by

$$\frac{\mathrm{d}\lambda^\mu{}_{(\alpha)}}{\mathrm{d}\tau} = \Phi_{(\alpha)}{}^{(\beta)}\lambda^\mu{}_{(\beta)}, \tag{3}$$

where $\Phi_{(\alpha)(\beta)}$ is an antisymmetric tensor by equation (2). This *acceleration tensor* has 'electric' and 'magnetic' components given respectively by $\boldsymbol{a}$ and $\boldsymbol{\omega}$ in close analogy with the Faraday tensor. That is, $\Phi_{(0)(i)} = a_i$ and $\Phi_{(i)(j)} = \epsilon_{ijk}\omega^k$, where the spacetime scalars $\boldsymbol{a}(\tau)$ and $\boldsymbol{\omega}(\tau)$ represent respectively the local translational acceleration, $a_i = A_\mu\lambda^\mu{}_{(i)}$, and the frequency of rotation of the local spatial frame of the observer with respect to a local nonrotating (i.e. Fermi–Walker transported) frame. We note that $\Phi^{\mu\nu} = \Phi^{(\alpha)(\beta)}\lambda^\mu{}_{(\alpha)}\lambda^\nu{}_{(\beta)}$, hence

$$\Phi^{\mu\nu} = A^\mu u^\nu - A^\nu u^\mu + \epsilon^{\mu\nu\rho\sigma}u_\rho\Omega_\sigma, \tag{4}$$

where $\Omega^\sigma = \omega^k\lambda^\sigma{}_{(k)}$ and $\epsilon_{0123} := 1$.

Let us now consider the most physically natural (Fermi) coordinate system in the neighbourhood of the worldline of the accelerated observer [5]. That is, we wish to establish a geodesic coordinate system along the path of the observer based on the tetrad $\lambda^\mu{}_{(\alpha)}$. At any given proper time τ, the straight spacelike geodesic lines normal to the observer's worldline span a Euclidean hyperplane. For a point on this hyperplane with inertial coordinates x^μ, let $X^\mu = (T, \boldsymbol{X})$ be the geodesic (Fermi) coordinates such that $X^0 = T = \tau$ and

$$x^\mu - \bar{x}^\mu(\tau) = X^i\lambda^\mu{}_{(i)}(\tau). \tag{5}$$

Differentiating this equation and using equation (3), we find

$$\mathrm{d}x^\mu = \left[P\lambda^\mu{}_{(0)} + Q^j\lambda^\mu{}_{(j)}\right]\mathrm{d}X^0 + \lambda^\mu{}_{(i)}\,\mathrm{d}X^i, \tag{6}$$

where P and Q are given by

$$P(T, \boldsymbol{X}) = 1 + \boldsymbol{a}(T)\cdot\boldsymbol{X}, \qquad \boldsymbol{Q}(T, \boldsymbol{X}) = \boldsymbol{\omega}(T)\times\boldsymbol{X}. \tag{7}$$

Thus the Minkowski metric $\mathrm{d}s^2 = \eta_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta$ can now be written in geodesic coordinates as $\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}X^\mu\,\mathrm{d}X^\nu$, where

$$g_{00} = -P^2 + Q^2, \qquad g_{0i} = Q_i, \qquad g_{ij} = \delta_{ij}. \tag{8}$$

It is simple to see that $\det(g_{\mu\nu}) = -P^2$ and

$$g^{00} = -\frac{1}{P^2}, \qquad g^{0i} = \frac{Q_i}{P^2}, \qquad g^{ij} = \delta_{ij} - \frac{Q_i Q_j}{P^2}. \tag{9}$$

The observer $\mathcal{O}$ occupies the spatial origin of the Fermi coordinates, which are admissible for $g_{00} < 0$. In this case, the domain of admissibility has been investigated in [6].

The connection coefficients are evaluated using equations (8) and (9). The only nonzero Christoffel symbols are given by

$$\Gamma^0_{00} = \frac{\boldsymbol{S}\cdot\boldsymbol{X}}{P}, \qquad \Gamma^0_{0i} = \frac{a_i}{P}, \qquad \Gamma^i_{0j} = -\left(\epsilon_{ijk}\omega^k + \frac{Q_i a_j}{P}\right), \tag{10}$$

$$\Gamma^i_{00} = P a_i - \frac{\boldsymbol{S}\cdot\boldsymbol{X}}{P} Q_i + [\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{X}) + \dot{\boldsymbol{\omega}}\times\boldsymbol{X}]_i. \tag{11}$$

Here an overdot denotes differentiation with respect to T and we have introduced the vector

$$\boldsymbol{S}(T) := \dot{\boldsymbol{a}} + \boldsymbol{a}\times\boldsymbol{\omega}. \tag{12}$$

It is important to observe here that $J^\mu := \mathrm{d}A^\mu/\mathrm{d}\tau$ and $\Sigma^\mu := \mathrm{d}\Omega^\mu/\mathrm{d}\tau$ are given in the local tetrad frame by $J^{(\alpha)} = (\boldsymbol{a}\cdot\boldsymbol{a}, \boldsymbol{J})$ and $\Sigma^{(\alpha)} = (\boldsymbol{a}\cdot\boldsymbol{\omega}, \dot{\boldsymbol{\omega}})$, where $\boldsymbol{J} = \dot{\boldsymbol{a}} + \boldsymbol{\omega}\times\boldsymbol{a}$.

The free particle $\mathcal{A}$ follows a geodesic in the new coordinate system. To express its motion relative to the reference observer $\mathcal{O}$, we need the reduced geodesic equation [1] in the new coordinate system

$$\frac{\mathrm{d}^2 X^i}{\mathrm{d}T^2} - \left(\Gamma^0_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}T}\frac{\mathrm{d}x^\beta}{\mathrm{d}T}\right)\frac{\mathrm{d}X^i}{\mathrm{d}T} + \Gamma^i_{\alpha\beta}\frac{\mathrm{d}X^\alpha}{\mathrm{d}T}\frac{\mathrm{d}X^\beta}{\mathrm{d}T} = 0, \tag{13}$$

where the Christoffel symbols are given by equations (10) and (11). The result is

$$\frac{\mathrm{d}^2\boldsymbol{X}}{\mathrm{d}T^2} + 2\boldsymbol{\omega}\times\boldsymbol{V} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{X}) + \dot{\boldsymbol{\omega}}\times\boldsymbol{X} + P\boldsymbol{a} - \frac{1}{c^2 P}(\boldsymbol{Q}+\boldsymbol{V})(\boldsymbol{S}\cdot\boldsymbol{X} + 2\boldsymbol{a}\cdot\boldsymbol{V}) = 0, \tag{14}$$

where $\boldsymbol{V} = \dot{\boldsymbol{X}}$ and $P = 1 + \boldsymbol{a}\cdot\boldsymbol{X}/c^2$. Here the presence of the speed of light c has been made explicit so that relativistic corrections to the Newtonian inertial accelerations can easily be identified.

Equation (14) contains the fully relativistic inertial accelerations of a free particle with respect to an accelerating and rotating reference system and has been derived in various forms by a number of authors (see [7–12] and the references cited therein). In the case of pure rotation ($\boldsymbol{a} = 0$), the inertial accelerations in equation (14) are just as in Newtonian mechanics. We note that a critical speed of $c/\sqrt{2}$ has appeared in the treatment of the motion of relativistic charged particles in the field of rotating magnetic lines of force corresponding to pulsar magnetospheres [13–15]. This critical speed turns out to be due to the particular mechanical model of the electromagnetic system discussed in [13]. In the theory of relativity, there is no critical speed associated with rotation per se as is evident from equation (14). The situation is different in the case of translational acceleration; however, before we turn to the case of purely translational accelerations, let us note the existence of relativistic inertial accelerations in equation (14) that are due to the coupling of acceleration and rotation.

Set $\boldsymbol{\omega} = 0$ and note that equation (14) reduces to

$$\frac{\mathrm{d}^2\boldsymbol{X}}{\mathrm{d}T^2} + P\boldsymbol{a} - \frac{\boldsymbol{V}}{c^2 P}(\dot{\boldsymbol{a}}\cdot\boldsymbol{X} + 2\boldsymbol{a}\cdot\boldsymbol{V}) = 0. \tag{15}$$

In the Newtonian limit ($c \to \infty$), this equation reduces to a standard result: from the viewpoint of the observer, the free particle has acceleration $-\boldsymbol{a}$. Writing $\boldsymbol{a}$ in equation (15) in terms

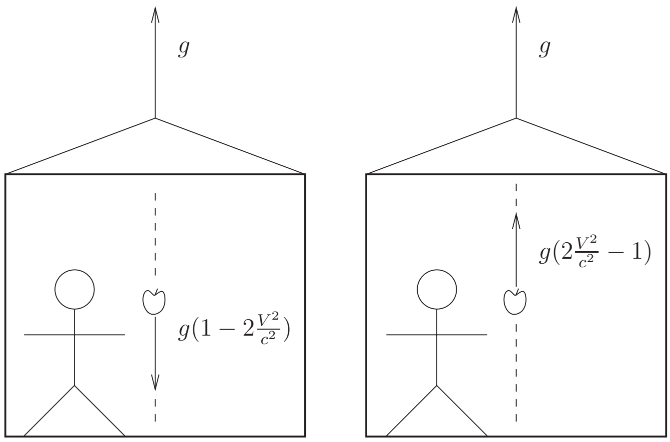

Figure 1. The inertial acceleration of a free particle according to the accelerated observer. In the left panel $|V| < c/\sqrt{2}$ and in the right panel $|V| > c/\sqrt{2}$.

of its components parallel and perpendicular to the instantaneous direction of the velocity $\boldsymbol{V} = V\widehat{\boldsymbol{V}}$, $\boldsymbol{a} = a_{\parallel}\widehat{\boldsymbol{V}} + \boldsymbol{a}_{\perp}$, the parallel component of the inertial acceleration in equation (15) to first order in $\boldsymbol{a}$ and neglecting $\dot{\boldsymbol{a}}$ is given by $-a_{\parallel}(1 - 2V^2/c^2)\widehat{\boldsymbol{V}}$. To illustrate this result, it is useful to assume one-dimensional motion of $\mathcal{A}$ along the Z-direction such that $\boldsymbol{a} = a(T)\widehat{\boldsymbol{Z}}$. In this case, to first order in a and neglecting $\dot{a}$, the inertial acceleration is simply given by $-a(1 - 2V^2/c^2)$. Without these approximations, the acceleration of the free particle relative to the observer at its position ($\boldsymbol{X} = 0$) is

$$\left.\frac{\mathrm{d}^2 Z}{\mathrm{d}T^2}\right|_{Z=0} = -a\left(1 - 2\frac{V^2}{c^2}\right), \tag{16}$$

so that if $|V| < V_c = c/\sqrt{2}$, the free particle has the expected direction of acceleration as observed by $\mathcal{O}$. On the other hand, for $|V| > V_c$ the inertial acceleration of $\mathcal{A}$ reverses direction. This situation is schematically illustrated in figure 1. If $|V| = V_c$, $\mathcal{A}$ has no inertial acceleration at $Z = 0$ [10].

Let us recall that in the standard discussion of Einstein's heuristic principle of equivalence, the observer at rest in the 'elevator' that is accelerated with acceleration $\boldsymbol{a}$ determines the acceleration of a free particle $\mathcal{A}$ ('apple') that moves past the observer along the same line. The standard discussion is limited to the Newtonian limit and in that limit the acceleration of $\mathcal{A}$ is $-\boldsymbol{a}$. However, if relativistic effects are taken into account, then the correct answer is given by equation (16). For relative speed above $V_c = c/\sqrt{2}$, the direction of acceleration is opposite to the result of Newtonian mechanics, which is counterintuitive, since our intuition is based on Newtonian expectations.

To clarify these issues further, let us consider a uniformly accelerated observer in hyperbolic motion with acceleration $g > 0$ moving along the positive z-direction in the background global inertial frame. The worldline of the observer is given by

$$\bar{t} = \frac{1}{g}\sinh g\tau, \qquad \bar{x} = \bar{y} = 0, \qquad \bar{z} = z_0 + \frac{1}{g}(-1 + \cosh g\tau), \tag{17}$$

so that at $\bar{t} = \tau = 0$, the observer is at rest at $\bar{z} = z_0$. The natural nonrotating tetrad frame along the observer's worldline has nonzero components $\lambda^0{}_{(0)} = \lambda^3{}_{(3)} = \gamma$, $\lambda^3{}_{(0)} = \lambda^0{}_{(3)} = \beta\gamma$

and $\lambda^1{}_{(1)} = \lambda^2{}_{(2)} = 1$, where $\beta = \tanh g\tau$ and $\gamma = \cosh g\tau$. It follows from equation (5) that the inertial coordinates are related to the coordinates of the accelerated (geodesic) frame via

$$t = \left(Z + \frac{1}{g}\right) \sinh gT, \tag{18}$$

$$x = X, \qquad y = Y, \tag{19}$$

$$z = z_0 - \frac{1}{g} + \left(Z + \frac{1}{g}\right) \cosh gT, \tag{20}$$

so that at $X^\mu = (\tau, \mathbf{0})$, equations (18)–(20) reduce to equation (17). The new (Rindler) coordinates are $T, X, Y \in (-\infty, \infty)$ and $Z \in (-1/g, \infty)$, where $Z = -1/g$ is the Rindler horizon and corresponds to a null cone in the inertial frame, since $(Z + 1/g)^2 = (z - z_0 + 1/g)^2 - t^2$.

The free particle follows a straight line in the inertial frame and this fact can be used to find an explicit solution to equation (15) in this case. We limit our discussion to motion of the free particle along the z-direction

$$z = v_0 t + z_0, \tag{21}$$

such that at $t = 0$ it passes the observer with relative speed v_0. It follows from equations (18)–(20) that

$$Z = \frac{1}{g}\left(-1 + \frac{1}{\cosh gT - v_0 \sinh gT}\right). \tag{22}$$

Computing the inertial acceleration of the free particle with respect to the observer, we find that $\ddot{Z}(T = 0) = -g\left(1 - 2v_0^2\right)$, where $v_0 = \dot{Z}(T = 0)$. This is the same result that we obtained before in equation (16) for the general case of variable acceleration.

It is straightforward to extend our analysis of the motion of $\mathcal{A}$ relative to $\mathcal{O}$ to the curved spacetime of general relativity along the lines indicated in [8, 9, 16]. The origin of the interesting factor $(1 - 2V^2/c^2)$ turns out to be the same for both tidal accelerations as well as the translational inertial acceleration: it comes about in the transition from the standard geodesic equation to its reduced form (13). That is, the factor $(1 - 2V^2/c^2)$ is basically due to the representation of the motion of $\mathcal{A}$ in terms of the proper time of the observer $\mathcal{O}$ rather than the proper time of the free particle $\mathcal{A}$.

The physical phenomena associated with the factor $(1 - 2V^2/c^2)$ in equations (15) and (16) come about only when the motion of $\mathcal{A}$ is referred to Fermi coordinates; that is, they do not in general appear in any other coordinate system. Nevertheless, equations (15) and (16) are constructed from scalar invariants and thus express real physically measurable effects. This circumstance may be illustrated as follows: the magnitude of the solution of equation (15), $|\mathbf{X}(T)|$, is the proper distance between the worldlines of $\mathcal{O}$ and $\mathcal{A}$, measured along a spacelike geodesic that is normal to the worldline of the observer $\mathcal{O}$ at its proper time T. As such, this invariant quantity can be computed in any admissible system of coordinates. Fermi coordinates are advantageous in practice as they are the most physically natural system of coordinates; therefore, equations (15) and (16) can be subjected to *direct* experimental test if the observer employs a local coordinate system that closely approximates a Fermi system.

Finally, let us remark that the proper critical speed $c/\sqrt{2}$ has also been discussed for one-dimensional relative motion along the radial direction in the context of the exterior Schwarzschild geometry in [17].

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