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LETTER TO THE EDITOR

High-order corrections to the entropy and area of quantum black holes

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Abstract

The celebrated area–entropy formula for black holes has provided the most important clue in the search for the elusive theory of quantum gravity. We explore the possibility that the (linear) area–entropy relation acquires some smaller corrections. Using the Boltzmann–Einstein formula, we rule out the possibility for a power-law correction and provide severe constraints on the coefficient of a possible log-area correction. We argue that a non-zero logarithmic correction to the area–entropy relation would also imply a modification of the area–mass relation for quantum black holes.

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The necessity for a quantum theory of gravity has been recognized since the 1930s. However, despite the flurry of research we still lack a complete theory of quantum gravity. It is believed that black holes may play a major role in our attempts to shed some light on the nature of a quantum theory of gravity (such as the role played by atoms in the early development of quantum mechanics).

In particular, the area–entropy relation $S_{\text{BH}} = A/4\ell_P^2$ [1] for black holes has served as a valuable element of guidance for quantum-gravity research. The intuition that led Bekenstein to this discovery is actually based on very simple ingredients. In particular, to elucidate the relation between area and entropy, it is instructive to use a semiclassical version of Christodoulou’s reversible processes [2, 3], in which a particle is absorbed by a black hole. Bekenstein [1, 4] has shown that the *Heisenberg quantum uncertainty principle* imposes a lower bound on the increase in black-hole surface area:

$$(\Delta A)_{\text{min}} = \gamma \ell_P^2, \quad (1)$$

where γ is a dimensionless constant of order unity and $\ell_P = \left(\frac{G}{c^3}\right)^{1/2} \hbar^{1/2}$ is the Planck length. Remarkably, this bound is *universal* in the sense that it is *independent* of the black-hole

parameters¹. The universality of the fundamental lower bound is clearly strong evidence in favour of a *uniformly* spaced area spectrum for quantum black holes (see [6]). Hence, one concludes that the quantization condition of the black-hole surface area should be of the form

$$A_n = \gamma \ell_p^2 n, \quad n = 1, 2, \dots, \quad (2)$$

Furthermore, using the fact that the minimum increase in the black-hole surface area should correspond to a minimum increase of its entropy (in order to compensate for the loss of the particle entropy), one arrives at the *proportionality* between black-hole surface area and entropy $S_{\text{BH}} = \eta A / \ell_p^2$.

It should be recognized, however, that the precise values of the proportionality constants γ and η cannot be inferred from this simple line of reasoning. The very nature of the Heisenberg quantum uncertainty principle allows only an order-of-magnitude estimate of the minimal increase in black-hole surface area. As a consequence, the proportionality constant η was fixed only few years later by Hawking, who determined the characteristic temperature of black holes [7].

Mukhanov and Bekenstein [6, 8, 9] have suggested an independent argument in order to determine the value of the coefficient γ . In the spirit of the Boltzmann–Einstein formula in statistical physics, they relate $g_n \equiv \exp[S_{\text{BH}}(n)]$ to the number of microstates of the black hole that correspond to a particular external macrostate. In other words, g_n is the degeneracy of the n th area eigenvalue. The thermodynamic relation between black-hole surface area and entropy can be met with the requirement that g_n has to be an integer for every n only when

$$\gamma = 4 \ln k, \quad (3)$$

where k is some natural number. Thus, statistical physics arguments force the dimensionless constant γ in equation (2) to be of the form of equation (3).

Nevertheless, a specific value of k requires further input. This information may emerge by applying *Bohr's correspondence principle* [10] to the (discrete) quasinormal frequencies of black holes [11, 12]. This argument provides the missing link, and gives evidence in favour of the value $k = 3$. It should be mentioned that following the pioneering work of Bekenstein [1], a number of independent calculations (most of them in the last few years) have indicated that a black-hole surface area has a *discrete* spectrum. Moreover, many of them have recovered the uniformly spaced area spectrum equation (2) [13–20]. However, there is no general agreement on the spacing of the levels. The relation $\gamma = 4 \ln 3$ is the *unique* value consistent both with the area–entropy *thermodynamic* relation, with *statistical physics* arguments (namely, the Boltzmann–Einstein formula), and with *Bohr's correspondence principle*.

Moreover, we would like to emphasize that using Bohr's correspondence principle allows one to fix not only the value of k , but also to obtain the factor of 1/4 in the linear area–entropy relation (a factor which could not be fixed from Bekenstein's intuitive argument [1], and which was derived only later from Hawking's analysis of black-hole radiation [7]).

While it is well established that the leading term in the area–entropy relation is linear in the black-hole surface area, in recent years evidence has been mounting that smaller correction terms may also exist (see, e.g., [21] and references therein). These indications for sub-leading

¹ The universal lower bound equation (1) derived by Bekenstein is valid only for *neutral* particles [1, 4]. The assimilation of a quantum *charged* particle by a black hole yields a similar lower bound for the increase in black-hole surface area $(\Delta A)_{\text{min}} = 4\ell_p^2$ [5]. As was noted by Bekenstein [1, 4] (for neutral particles), the underlying physics which excludes a completely reversible process is the *Heisenberg quantum uncertainty principle*. However, for *charged* particles it must be supplemented by another physical mechanism [5], a Schwinger discharge of the black hole (*vacuum polarization* effects). Without this physical mechanism, one could have reached the reversible limit ($\Delta A = 0$). It is remarkable that the lower bound found for charged particles is of the same order of magnitude as that given by Bekenstein for neutral particles, even though they emerge from *different* physical mechanisms. This is clearly strong evidence in favour of a *uniformly* spaced area spectrum for quantum black holes.

logarithmic terms have arisen both in string theory and in loop quantum gravity. One should emphasize that there is, however, *no* general agreement on the *coefficient* of the logarithmic correction [21]. It is therefore very important to establish constraints on the possible values that these sub-leading entropy corrections may take. To that end, we consider a general area–entropy relation for black holes of the form

$$S_{bh} = \frac{1}{4} \frac{A}{\ell_p^2} + \alpha_1 \left(\frac{A}{\ell_p^2} \right)^\beta + \ln[f(A)], \quad (4)$$

where $\beta > 0$, and $f(A)$ is a function of the black-hole surface area. In addition, we write the quantized black-hole surface area in the form

$$A_n = (\gamma_0 n + \gamma_1 n^\delta + \gamma_2 \ln n) \ell_p^2, \quad (5)$$

where $n = 1, 2, \dots$,² For g_n to be an integer for every n , $f(A)$ should equal a natural number for every A_n , implying that $f(A)$ should be of the form $f(A) = \sum_{j=0}^{j_{\max}} c_j A^j$, such that all powers in the series are natural numbers. In the large area limit we expand $\ln[f(A)] = \alpha_2 \ln A + \alpha_3 + \dots$. Substituting equation (5) in equation (4), and using the Boltzmann–Einstein relation $g_n = \exp[S_{bh}(n)]$ (with the requirement that g_n is an integer for every n), one obtains severe constraints on the possible values that the various coefficients may take. First, the Boltzmann–Einstein formula implies that γ_0 should be of the form [6, 8–10]

$$\gamma_0 = 4 \ln k, \quad (6)$$

where k is a natural number. In addition, we find

$$\alpha_1 = \gamma_1 = 0. \quad (7)$$

Thus, our simple argument implies that there are *no stronger-than-logarithmic* corrections to the area–entropy relation. To continue, one has to consider two distinct cases:

Case I. Non-vanishing logarithmic corrections to the area–entropy relation ($\alpha_2 \neq 0$). In this case one finds that the coefficients must satisfy the constraints

$$\gamma_2 = 0, \quad (8)$$

and

$$\alpha_2 = l, \quad \alpha_3 = \ln \left(\frac{m}{\gamma_0^l} \right), \quad (9)$$

where l and m are natural numbers. Thus, the black-hole surface area takes a very simple form

$$A_n = 4\ell_p^2 \ln k \cdot n, \quad n = 1, 2, \dots \quad (10)$$

Such a *uniformly* spaced area spectrum (with *no* sub-leading corrections) supports the existence of a *fundamental* area unit³. In addition, the area–entropy relation should be of the restricted

² It is possible to add a constant term in equation (5) as well. It would then be straightforward to use the Boltzmann–Einstein formula in order to derive similar constraints on the possible values that this constant term may take.

³ Analysing carefully the Gedanken experiment of Bekenstein [1], or the wave analysis of [10], one concludes that $\Delta A \sim \ell_p^2 + O(\frac{\ell_p^4}{M^2})$. The sub-leading term is a consequence of two distinct factors: (i) the second term in the area–mass relation $\Delta A = 32\pi M \Delta M + 16\pi (\Delta M)^2$ with $\Delta M \sim \frac{\ell_p^2}{M}$, and (ii) the gravitational back-reaction caused by the particle energy $E \sim \frac{\ell_p^2}{M}$ (in Bekenstein’s analysis), or the gravitational wave energy in [10], which would change the effective black-hole mass in these analyses from M to $M + O(\frac{\ell_p^2}{M})$. The relation $\Delta A \sim \ell_p^2 + O(\frac{\ell_p^4}{M^2})$ suggests that A_n acquires a sub-leading correction term $O(\ln n)$. We have learned, however, that *if* a logarithmic correction exists in the area–entropy relation, then the area spectrum should be uniformly spaced. This may indicate that the area–mass relation for quantum black holes should also acquire higher-order corrections of the form $A = 16\pi M^2 + \xi \ell_p^2 \ln(\frac{M^2}{\ell_p^2})$, where ξ is a dimensionless constant of order unity. Such a relation may allow $\Delta A \sim \ell_p^2$ without $O(\frac{\ell_p^4}{M^2})$ corrections.

form

$$S_{bh} = S_{\text{BH}} + l \ln \left(\frac{A}{\ell_p^2} \right) + \ln \left[\frac{m}{(4 \ln k)^l} \right]. \quad (11)$$

The main conclusion is that the coefficient of the log-area correction should be a *natural* number.

Case II. No logarithmic corrections to the area–entropy relation ($\alpha_2 = 0$). In this case one finds that the coefficients must satisfy the constraints

$$\gamma_2 = 4l, \quad \alpha_3 = \ln m, \quad (12)$$

where l and m are natural numbers.

In summary, using a simple argument based on the Boltzmann–Einstein formula, we have derived severe constraints on the possible sub-leading corrections to the (semiclassical) Bekenstein–Hawking area–entropy relation for black holes. In particular, we have ruled out the possibility for a power-law correction, and found that the coefficient of a possible logarithmic correction should be a natural number.

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