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LETTER TO THE EDITOR

A note on Lie–Lorentz derivatives

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Abstract

The definition of 'Lie derivative' of spinors with respect to Killing vectors is extended to all kinds of Lorentz tensors. This Lie–Lorentz derivative appears naturally in the commutator of two supersymmetry transformations generated by Killing spinors and vanishes for vielbeins. It can be identified as the generator of the action of isometries on supergravity fields and its use for the calculation of supersymmetry algebras is revised and extended.

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1. Introduction

Spinors are defined by their transformation properties under $SO(n_+, n_-)$ ('Lorentz') transformations and, thus, can only be introduced in the tangent space of curved spaces using the formalism introduced by Weyl in [1]. This formalism makes use of vielbeins $\{e_a{}^\mu\}$ which form an orthonormal basis in tangent space

$$e_a{}^\mu e_b{}^\nu g_{\mu\nu} = \eta_{ab}, \qquad (\eta_{ab}) = \mathrm{diag}\,(+\cdots+, -\cdots-), \tag{1}$$

and it is covariant with respect to local transformations that preserve this orthonormality: local $SO(n_+, n_-)$ transformations. These are the only transformations that act non-trivially on any Lorentz tensor[1] T in the representation r,

$$T'(x') = \Gamma_r[g(x)]T(x), \tag{2}$$

where $\Gamma_r[g(x)]$ is the representation of the position-dependent group element $g(x)$ that can be constructed by exponentiation

$$\Gamma_r[g(x)] = \exp\left[\tfrac{1}{2}\sigma^{ab}(x)\Gamma_r(M_{ab})\right], \tag{3}$$

[1] We will call Lorentz tensor any object transforming in some finite-dimensional representation of the Lorentz group, like vectors and spinors.

where Γ_r (M_{ab}) are the $SO(n_+, n_-)$ generators in the representation r. For the contravariant vector and spinor representations

$$\Gamma_v(M_{ab})^c{}_d = 2\eta_{[a}{}^c\eta_{b]d}, \qquad \Gamma_s(M_{ab}) = \tfrac{1}{2}\Gamma_{[a}\Gamma_{b]}, \qquad \{\Gamma_a, \Gamma_b\} = 2\eta_{ab}. \tag{4}$$

Local Lorentz covariance is required in order to gauge away the additional degrees of freedom that the vielbein has (d^2), as compared to the metric ($d(d+1)/2$), and it is achieved by introducing a covariant derivative $\mathcal{D}_\mu$ which acts on T according to

$$\mathcal{D}_\mu T \equiv (\partial_\mu - \omega_{r\mu})T, \qquad \omega_{r\mu} \equiv \tfrac{1}{2}\omega_\mu{}^{ab}\Gamma_r(M_{ab}). \tag{5}$$

$\omega_\mu{}^{ab}$ is the $SO(n_+, n_-)$ (spin) connection, i.e., it is antisymmetric $\omega_\mu{}^{ab} = -\omega_\mu{}^{ba}$, which implies

$$\mathcal{D}_\mu \eta_{ab} = 0 \tag{6}$$

and transforms according to

$$\omega'_{r\mu} = \Gamma_r[g(x)]\omega_{r\mu}\Gamma_r^{-1}[g(x)] + (\partial_\mu\Gamma_r[g(x)])\Gamma_r^{-1}[g(x)]. \tag{7}$$

In order to have only one connection, the spin connection is related to the affine connection $\Gamma_{\mu\nu}{}^\rho$ by the vielbein postulate

$$\nabla_\mu e_a{}^\nu = \partial_\mu e_a{}^\nu + \Gamma_{\mu\rho}{}^\nu e_a{}^\rho - e_b{}^\nu\omega_{\mu a}{}^b = 0 \tag{8}$$

(∇_μ denotes the total (general and Lorentz) covariant derivative), which implies

$$\omega_{\mu a}{}^b = \Gamma_{\mu a}{}^b - e_a{}^\nu\partial_\mu e_\nu{}^b. \tag{9}$$

The vielbein postulate together with equation (6) implies that the affine connection is metric compatible and the spin connection is completely determined by the vielbein, their inverses, and the contorsion tensor K_{abc}:

$$\omega_{abc} = -\Omega_{abc} + \Omega_{bca} - \Omega_{cab} + K_{abc}, \qquad \Omega_{abc} = e_a{}^\mu e_b{}^\nu\partial_{[\mu|}e_{c|\nu]}. \tag{10}$$

In Weyl's formalism Lorentz tensors (in particular spinors) behave, then, as scalars under general coordinate transformations (GCTs). However, we can use Weyl's formalism just to work in curvilinear coordinates in Minkowski spacetime and we would find that a standard Lorentz transformation becomes a GCT that does not act on the spinorial indices. This looks strange, but it is unavoidable in Weyl's formalism.

On the other hand, Lorentz transformations and GCTs do not commute and the result of a GCT on a Lorentz tensor is strongly frame-dependent. This shows up in the standard Lie derivative (an infinitesimal GCT) which does not transform Lorentz tensors into Lorentz tensors. Thus, while the Lie derivative of the metric $g_{\mu\nu}$ with respect to a Killing vector field vanishes (by definition) the Lie derivative of the inverse vielbein $e^a{}_\mu$ associated with the same metric with respect to the same Killing vector in general does not.

For spinors ψ this situation was solved in [3][2] where a *spinorial Lie derivative* $\mathbb{L}_k$ with respect to Killing vector fields k was defined by[3]

$$\mathbb{L}_k\psi \equiv k^\mu\mathcal{D}_\mu\psi + \tfrac{1}{4}\mathcal{D}_{[a}k_{b]}\Gamma^{ab}\psi. \tag{11}$$

This derivative is a derivation that transforms spinors into spinors (it is Lorentz-covariant) and satisfies the property

$$\left[\mathbb{L}_{k_1}, \mathbb{L}_{k_2}\right]\psi = \mathbb{L}_{[k_1,k_2]}\psi, \tag{12}$$

[2] For a different approach and further references, see, e.g. [4]. The Lie derivative on spinors has been studied and used in the derivation of supersymmetry algebras in [5, 6] and more recently in [7] whose results are to be revised and extended.

[3] We use the symbol $\mathbb{L}_k$ to avoid confusion with the standard Lie derivative that we keep denoting by $\mathcal{L}_k$. On spinors, then, $\mathcal{L}_k\psi = k^\mu\partial_\mu\psi$.

but, most importantly, the second term defines an action of certain GCTs (the isometries) on the spinorial indices: an infinitesimal Lorentz transformation with parameter $\frac{1}{2}\mathcal{D}_{[a}k_{b]}$. This is precisely what one expects on physical grounds. Let us consider an example: Minkowski spacetime $g_{\mu\nu} = \eta_{\mu\nu}$ with the obvious vielbein $e_a{}^\mu = \delta_a{}^\mu$ and the infinitesimal GCT generated by the Killing vector

$$k^\mu = \sigma^\mu{}_\nu x^\nu, \tag{13}$$

where $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$ and is constant. This is a standard infinitesimal Lorentz transformation and, indeed, we find

$$\mathbb{L}_k = k^\mu \partial_\mu \psi + \tfrac{1}{4}\sigma_{ab}\Gamma^{ab}\psi, \tag{14}$$

which is the result that one would obtain in the standard spinor formalism.

The spinorial Lie derivative can be seen as a Lorentz covariantization of the standard Lie derivative (first term) supplemented by an infinitesimal local Lorentz transformation that trivializes the holonomy. It is clear that these ideas can be extended to other Lorentz tensors and we can define a Lie–Lorentz derivative which is first a Lorentz covariantization of the standard Lie derivative supplemented by an infinitesimal Lorentz transformation that trivializes the holonomy. The parameter of this transformation has to be exactly the same as in the spinorial case and, thus, we arrive at the following definition.

2. Definition and properties of the Lie–Lorentz derivative

On pure Lorentz tensors T we define the Lie–Lorentz derivative with respect to the Killing vector k by

$$\mathbb{L}_k T \equiv k^\rho \nabla_\rho T + \tfrac{1}{2}\nabla_{[a}k_{b]}\Gamma_r(M^{ab})T. \tag{15}$$

On tensors that also have world indices $T_{\mu_1\cdots\mu_m}{}^{\nu_1\cdots\nu_n}$

$$\begin{aligned}\mathbb{L}_k T_{\mu_1\cdots\mu_m}{}^{\nu_1\cdots\nu_n} \equiv{}& k^\rho \nabla_\rho T_{\mu_1\cdots\mu_m}{}^{\nu_1\cdots\nu_n} - \nabla_\rho k^{\nu_1} T_{\mu_1\cdots\mu_m}{}^{\rho\nu_2\cdots\nu_n} - \cdots \\ &+ \nabla_{\mu_1}k^\rho T_{\rho\mu_2\cdots\mu_m}{}^{\nu_1\cdots\nu_n} + \cdots + \tfrac{1}{2}\nabla_{[a}k_{b]}\Gamma_r(M^{ab})T_{\mu_1\cdots\mu_m}{}^{\nu_1\cdots\nu_n}.\end{aligned} \tag{16}$$

In all the cases that we are going to consider, ∇_μ is the full (affine plus Lorentz) torsionless covariant derivative satisfying the vielbein postulate.

In the following T_1, T_2 will be two mixed tensors of any kind, k_1, k_2 any two conformal Killing vector fields and a^1, a^2 two arbitrary constants. The Lie–Lorentz derivative has the following basic properties.

1. Leibnitz rule:

$$\mathbb{L}_k(T_1T_2) = \mathbb{L}_k(T_1)T_2 + T_1\mathbb{L}_kT_2. \tag{17}$$

   Thus, it is a derivation.
2. The commutator of two Lie–Lorentz derivatives

$$\left[\mathbb{L}_{k_1}, \mathbb{L}_{k_2}\right]T = \mathbb{L}_{[k_1,k_2]}T, \tag{18}$$

   where $[k_1, k_2]$ is their Lie bracket.
3. The Lie–Lorentz derivative is linear in the vector fields

$$\mathbb{L}_{a^1k_1+a^2k_2}T = a^1\mathbb{L}_{k_1}T + a^2\mathbb{L}_{k_2}T, \tag{19}$$

   and, thus, the Lie–Lorentz derivative with respect to the conformal Killing vector fields forms a representation of the Lie algebra of conformal isometries of the manifold.

Some immediate consequences of the definition and basic properties are as follows.

1. The Lie–Lorentz derivative of the vielbein is

$$\mathbb{L}_k e^a{}_\mu = \frac{1}{d} \nabla_\rho k^\rho e^a{}_\mu, \tag{20}$$

and vanishes when k is a Killing vector. In this case, we have the desirable property

$$\mathbb{L}_k \xi^a = e^a{}_\mu \mathcal{L}_k \xi^\mu. \tag{21}$$

2. The Lie–Lorentz derivative of gamma matrices is zero:

$$\mathbb{L}_k \gamma^a = 0. \tag{22}$$

3. As a consequence of equations (17), (20) and (22), the Lie–Lorentz derivative with respect to Killing vectors preserves the Clifford action of vectors v on spinors ψ, $v \cdot \psi \equiv v_a \Gamma^a \psi = \not{v}\psi$:

$$[\mathbb{L}_k, \not{v}]\psi = [k, v] \cdot \psi. \tag{23}$$

4. Also for Killing vectors k it preserves the covariant derivative

$$[\mathbb{L}_k, \nabla_v]T = \nabla_{[k,v]} T. \tag{24}$$

5. All these properties imply that the Lie–Lorentz derivative with respect to Killing vectors preserves the supercovariant derivative of supergravity theories, at least in the simplest cases that we are going to examine next.

We stress that properties 3, 4, 5 are only valid for Killing (not just conformal Killing) vectors.

3. The Lie–Lorentz derivative and supersymmetry

The Lie–Lorentz derivative occurs naturally in supergravity theories.

To start with, let us consider the local on-shell supersymmetry algebra, in particular, the commutator of two infinitesimal, local supersymmetry transformations $\delta_Q(\epsilon)$ in $N = 1$, $d = 4$ supergravity:

$$\delta_Q(\epsilon) e^a{}_\mu = -\mathrm{i}\bar{\epsilon}\gamma\psi_\mu, \qquad \delta_Q(\epsilon)\psi_\mu = \mathcal{D}_\mu \epsilon, \tag{25}$$

which is usually written in the form

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = \delta_{\mathrm{gct}}(\xi) + \delta_{\mathrm{LL}}(\sigma) + \delta_Q(\epsilon), \tag{26}$$

where $\delta_{\mathrm{gct}}(\xi)$ is an infinitesimal general coordinate transformation with parameter

$$\xi^\mu = -\mathrm{i}\bar{\epsilon}_1 \gamma^\mu \epsilon_2, \tag{27}$$

and is given by $\delta_{\mathrm{gct}}(\xi) = -\pounds_\xi$, $\delta_{\mathrm{LL}}(\sigma)$ is an infinitesimal local Lorentz transformation with parameter

$$\sigma^{ab} = \xi^\nu \omega_\nu{}^{ab}, \tag{28}$$

and where

$$\epsilon = \xi^\mu \psi_\mu. \tag{29}$$

We are interested in obtaining the global superalgebra from the commutators of all the symmetry transformations of the theory. There is no unique superalgebra associated with a given supergravity theory; rather, there are superalgebras associated with given bosonic solutions of the supergravity theory and their (super) symmetries. Solutions with a high degree of (super)symmetry are usually considered vacua of the supergravity theory and their associated superalgebras are of special interest.

Let us then consider a given vacuum solution of the $N = 1, d = 4$ supergravity equations of motion admitting Killing spinors ε and Killing vectors k:

$$\mathcal{D}_\mu \varepsilon = 0, \qquad \nabla_{(\mu} k_{\nu)} = 0. \tag{30}$$

On this vacuum, the commutator equation (26) should reduce to the commutator of two supercharges $Q_{1,2} = \bar{\varepsilon}_{1,2} Q$ which should be proportional to translations $\sim k^a P_a$, where $k^a = -\mathrm{i}\bar{\varepsilon}_1 \gamma^a \varepsilon_2$ is a Killing vector if, as we have assumed, $\varepsilon_{1,2}$ are Killing spinors. However, the commutator equation (26) does not give that result if we naively interpret $\delta_{\mathrm{gct}}(k)$ as an infinitesimal translation since there is another term δ_{LL} whose meaning is unclear.

Observing that, actually, all the Killing vectors $k^a = -\mathrm{i}\bar{\varepsilon}_1 \gamma^a \varepsilon_2$ are covariantly constant, we can instead write

$$[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] = \delta_P(k), \qquad k^a = -\mathrm{i}\bar{\varepsilon}_1 \gamma^a \varepsilon_2, \tag{31}$$

where now we have identified the generator of the vacuum isometries acting on the supergravity fields

$$\delta_P(k) \equiv -\mathbb{L}_k. \tag{32}$$

This commutator has an immediate interpretation in terms of the global superalgebra. On the other hand, in this form, on account of equation (20), it is evident that the commutator of two supersymmetry transformations generated by two Killing spinors leaves the vierbein invariant, as it should be.

To check that these definitions and interpretations actually make sense, let us consider a more complicated case: gauged $N = 2$, $d = 4$ supergravity, whose supersymmetry transformation rules are

$$\delta_Q(\epsilon) e^a{}_\mu = -\mathrm{i}\bar{\epsilon}\gamma^a \psi_\mu, \qquad \delta_Q(\epsilon) A_\mu = -\mathrm{i}\bar{\epsilon}\sigma^2 \psi_\mu, \qquad \delta_Q(\epsilon)\psi_\mu = \tilde{\mathcal{D}}_\mu \epsilon, \tag{33}$$

where

$$\tilde{\nabla}_\mu = \hat{\mathcal{D}}_\mu + \mathrm{i}g A_\mu \sigma^2 + \tfrac{1}{4} \tilde{\not{F}} \gamma_\mu \sigma^2 \tag{34}$$

is the *supercovariant derivative* and

$$\hat{\mathcal{D}}_\mu = \mathcal{D}_\mu - \frac{\mathrm{i}g}{2}\gamma_\mu \tag{35}$$

is the $\mathrm{AdS}_4 \sim SO(2,3)$ covariant derivative. The commutator of two $N = 2$, $d = 4$ supersymmetry transformations is usually written in this form

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = \delta_{\mathrm{gct}}(\xi) + \delta_{\mathrm{LL}}(\sigma) + \delta_e(\Lambda) + \delta_Q(\epsilon), \tag{36}$$

where the parameter of the infinitesimal local Lorentz transformations is now

$$\sigma^{ab} = \xi^\mu \omega_\mu{}^{ab} - g\bar{\epsilon}_2 \gamma^{ab} \epsilon_1 - \mathrm{i}\bar{\epsilon}_2 (\tilde{F}^{ab} - \mathrm{i}\gamma_5 {}^\star\tilde{F}^{ab})\, \sigma^2 \epsilon_1, \tag{37}$$

and where $\delta_e(\Lambda)$ are $U(1)$ gauge transformations of the vector field and charged gravitino with parameter

$$\Lambda = -\mathrm{i}\bar{\epsilon}_2 \sigma^2 \epsilon_1 + \xi^\nu A_\nu. \tag{38}$$

It is easy to see that on a vacuum solution admitting Killing spinors ε and Killing vectors k:

$$\tilde{\nabla}_\mu \varepsilon = 0, \qquad \nabla_{(\mu} k_{\nu)} = 0. \tag{39}$$

The commutator equation (36) can be written in the form

$$[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] = \delta_P(k) + \delta_e(\chi), \qquad k^a = -\mathrm{i}\bar{\varepsilon}_1 \gamma^a \varepsilon_2, \qquad \chi = -\mathrm{i}\bar{\varepsilon}_2 \sigma^2 \varepsilon_1 + k^\nu A_\nu. \tag{40}$$

The interpretation is again immediate. This commutator should vanish on all the fields of the theory. It clearly does on the vierbein. On the vector and gravitino fields, it tells us that the Lie derivative vanishes up to a gauge transformation.

Let us now consider the remaining commutators.

Since we are identifying the bosonic generators of the bosonic subalgebra of the supersymmetry algebra of the vacuum with the Lie–Lorentz derivative with respect to the Killing vector fields of the solution, we are going to get as bosonic subalgebra the Lie algebra of isometries of the vacuum, on account of equation (18):

$$[\delta_P(k_1), \delta_P(k_2)] = \delta_P([k_1, k_2]). \tag{41}$$

The commutator of local supersymmetry transformations and GCTs are difficult to compute, as a matter of principle, since the standard Lie derivative of Lorentz tensors is not a Lorentz tensor and then it does not make sense to perform a further supersymmetry transformation on the transformed tensor. Further, while we have a prescription to calculate the effect of an infinitesimal GCT on any geometrical object, we do not know how to calculate the effect of an infinitesimal supersymmetry transformation of geometrical objects which are not fields of our theory (for instance, the vector that generates the infinitesimal GCT).

Thus, it is necessary to give a prescription to calculate these commutators. In [7] the following prescription was proposed:

$$[\delta_Q(\varepsilon), \delta_P(k)] = \delta_Q(\mathbb{L}_k\varepsilon). \tag{42}$$

This prescription is based on the property (checked for certain geometrical Killing spinors in that reference) that the Lie–Lorentz derivative preserves the supercovariant derivative and, therefore, transforms Killing spinors into Killing spinors.

This property also holds in the simple theories we are considering here: on account of equations (23),(24), we get

$$[\mathbb{L}_k, \hat{\mathcal{D}}_v]\epsilon = \hat{\mathcal{D}}_{[k,v]}\epsilon, \tag{43}$$

(so the AdS_4 covariant derivative is preserved and the Lie–Lorentz derivative transforms $N = 1$, AdS_4 Killing spinors into Killing spinors [7]) and, if

$$\mathbb{L}_k A_a = 0, \tag{44}$$

we find that the $N = 2$, AdS_4 supercovariant derivative is also preserved,

$$[\mathbb{L}_k, \tilde{\mathcal{D}}_v]\epsilon = \tilde{\mathcal{D}}_{[k,v]}\epsilon, \tag{45}$$

and the Lie–Lorentz derivative again transforms Killing spinors into Killing spinors.

Condition (44) is satisfied in most cases in which k is a Killing vector (up to a gauge transformation). If it was not satisfied in any gauge, the Killing vector k would be an isometry but not a symmetry of the complete supergravity background and would not be a generator of the vacuum supersymmetry algebra. Thus, it must always be satisfied.

It is clear that these results can be generalized to higher dimensions and supersymmetries.

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