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Spacetime Ehlers group: transformation law for the Weyl tensor

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Abstract

The spacetime Ehlers group, which is a symmetry of the Einstein vacuum field equations for strictly stationary spacetimes, is defined and analysed in a purely spacetime context (without invoking the projection formalism). In this setting, the Ehlers group finds its natural description within an infinite-dimensional group of transformations that maps Lorentz metrics into Lorentz metrics and which may be of independent interest. The Ehlers group is shown to be well defined independently of the causal character of the Killing vector (which may become null on arbitrary regions). We analyse which global conditions are required on the spacetime for the existence of the Ehlers group. The transformation law for the Weyl tensor under Ehlers transformations is obtained explicitly. This allows us to study where, and under what circumstances, curvature singularities in the transformed spacetime will arise. The results of the paper are applied to obtain a local characterization of the Kerr–NUT metric.

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1. Introduction

Strictly stationary spacetimes (i.e. spacetimes admitting a Killing vector which is timelike everywhere) can be conveniently studied by using the projection formalism introduced by Geroch [1], which consists in factoring out the action of the Killing field by projecting all spacetime objects onto the set of trajectories of the Killing vector. This method has been used extensively especially because Einstein's field equations simplify notably in this formalism (see [2] for a recent review on time-independent gravitational fields). It was used, for instance, to find that Einstein's vacuum field equations for stationary spacetimes admit a finite-dimensional symmetry group (i.e. a group of transformations that maps solutions into solutions). This is the so-called Ehlers group [1, 3], which has been applied to many problems, ranging from the discovery of new solutions (see, e.g., [4]) to the proof that the vacuum field equations in the stationary and axially symmetric case form an integrable system ([5] and, for example, [6] and references therein).

Despite the power of the projection formalism, there are circumstances in which it cannot be applied. For instance, the set of trajectories of the Killing vector may fail to be a smooth manifold. This is no problem when only local considerations are relevant because for any point in the spacetime, there always exists a sufficiently small neighbourhood of it such that the quotient set is a smooth manifold. However, for problems involving global aspects the question does become important and one should investigate whether the set of trajectories is a smooth manifold. This is, in general, difficult (see, however, [7] for a set of necessary conditions for the quotient to be a smooth manifold). A second, and perhaps more important, shortness of the projection formalism is that stationary spacetimes may develop ergospheres and horizons. At points where the Killing vector becomes null, the metric in the manifold of trajectories becomes degenerate and the projection formalism cannot be used. This makes this formalism unsuitable for studying most of the problems concerning stationary rotating black holes (in particular, their uniqueness properties) because the whole domain of outer communication cannot be covered by the method. A similar problem arises for some rapidly rotating objects, which may also have an ergosphere in their exterior (a rotating disc of dust [8], for instance, shows this behaviour).

Thus, some other method must be used to analyse problems in which the projection formalism cannot be used. A natural approach is to work directly on the spacetime and use only spacetime objects. In general, this is more difficult because the action of the Killing vector has not been factored out and the very existence of the Killing vector still needs to be imposed on the spacetime. Nevertheless, this method can be used in all situations where the projection formalism fails. Furthermore, working directly on the spacetime sometimes gives new insights into the problem. This method have been used recently to obtain local [9] and semilocal [10] characterizations of the Kerr metric which hold everywhere (including the ergorsphere and/or the black hole region) and which involve spacetime objects only.

The Ehlers group mentioned above was defined within the projection formalism and is known to map locally a strictly stationary vacuum solution into another strictly stationary vacuum solution (locally means that there exists a suitably small open neighbourhood of any point where the Ehlers transformation can be defined). This is generally sufficient for generating new vacuum solutions, because one can apply the transformation locally and, if desired and possible, the transformed spacetime can be extended to a maximal solution. However, there are other problems in which understanding the global properties of the action of the Ehlers group is important. For instance, there are approaches [11, 12] for proving uniqueness theorems for stationary black holes which make use of the Ehlers transformation (and its generalization to other nonlinear sigma models). However, this can only be justified as long as the global properties of the Ehlers transformation can be controlled. Thus, studying in detail the global requirements for the existence of the Ehlers transformation becomes necessary. As discussed above, the projection formalism is not suitable to analyse this kind of problem. Furthermore, the Ehlers group is defined only on regions where the Killing vector is timelike or spacelike and it is not clear *a priori* whether it can be smoothly extended through ergospheres or horizons. Explicit examples suggest that this extension can be performed, but no general proof has been given. To answer these questions we need to define and analyse the Ehlers group within a framework that avoids using the projection formalism.

In this paper we perform a detailed study of the Ehlers transformation in a spacetime setting. This will allow us to prove, first of all, that the Ehlers transformation is well defined at points where the Killing vector is null, as one could have expected. More interestingly, the spacetime approach will reveal several properties of the Ehlers group which are hidden in the quotient description. In particular, we will show that the Ehlers group finds its natural description within an infinite-dimensional group of transformations which maps Lorentzian metrics into Lorentzian metrics. The general form of the transformation is

$$g'_{\alpha\beta} \equiv \Omega^2 g_{\alpha\beta} - \zeta_\alpha W_\beta - \zeta_\beta W_\alpha - \lambda\Omega^{-2} W_\alpha W_\beta, \tag{1}$$

where ζ^α is an arbitrary vector field, $\lambda = -\zeta^\alpha\zeta_\alpha$ and W_α is an arbitrary 1-form constrained to satisfy $\Omega^2 \equiv 1+\zeta^\alpha W_\alpha > 0$. The transformation (1) includes as particular cases the Kerr–Schild transformation [13] (by setting $W_\alpha \propto \zeta_\alpha$ and $\zeta^\alpha\zeta_\alpha = 0$ everywhere) and a transformation put forward and studied by Bonanos [14] (when $\zeta^\alpha W_\alpha = 0$ and hence $\Omega^2 = 1$). Since (1) contains the Kerr–Schild transformation as a particular case, it also allows for a generalization of Kerr–Schild symmetries, which have been defined and studied recently in [15]. This issue, however, will not be considered further in this paper.

The set of transformations (1) will be shown to form a group and its basic properties will be discussed. Obviously, this full group does not map, in general, vacuum solutions into vacuum solutions, but it is likely that suitable subsets of it (besides the Ehlers group) do have this property. Using the spacetime description will allow us to discuss the necessary and sufficient conditions for the Ehlers transformation to be defined globally. Related to this question, the existence and location of curvature singularities in the transformed spacetime will be studied. This will be done by obtaining an *explicit* expression for the Weyl tensor of the transformed spacetime in terms of the original one. The transformation law turns out to be surprisingly simple and clear. Thus, the full geometry of the transformed metric can be determined without having to perform the Ehlers transformation explicitly. This may be particularly interesting for stationary and axially symmetric spacetimes where the Ehlers group extends to an infinite-dimensional group, the so-called Geroch group, which can be understood as an iteration of Ehlers transformations with respect to different Killing vectors. The results of this paper can also be applied to that situation.

The transformation law for the Weyl tensor will be applied to find a local characterization of the Kerr–NUT spacetime. First, we shall obtain the simplest subset of a vacuum solution which is invariant under Ehlers transformations. Its defining property turns out to be closely connected to the characterization of the Kerr metric found in [9]. This indicates once again that the Kerr metric enjoys a very privileged geometric position because it can be characterized by a property which also arises naturally from the Ehlers group (and therefore directly from the underlying structure of the Einstein vacuum field equations with a Killing vector).

The paper is organized as follows. In section 2 we write down several identities which are valid for any spacetime admitting a Killing vector, irrespective, of its causal character. They are useful for any four-dimensional description of spacetimes with a Killing vector (no field equations are assumed in this section). While some of these equations are well known, others appear to be new. In section 3 we introduce an infinite-dimensional group of transformations which maps Lorentzian metrics into Lorentzian metrics and we discuss its basic properties. In section 4 we introduce the Ehlers group as a particular case of this infinite-dimensional group of transformations. Then, we discuss what the requirements are for the Ehlers transformation to exist globally and we prove that vacuum solutions are mapped into vacuum solutions irrespective, of the causal character of the Killing vector. This shows that the Ehlers transformation is a symmetry of the vacuum field equations independently of whether the Killing vector has ergoregions and/or horizons. In section 5 we make use of the identities in section 2 in order to obtain the transformation law for the Weyl tensor under Ehlers transformations. The result is surprisingly simple and elegant. Having obtained the form of the transformed Weyl tensor, we can identify where and under what circumstances curvature singularities in the transformed spacetime will occur.

Finally, in section 6 we identify the simplest subset of stationary vacuum solutions which are invariant under Ehlers transformations. We classify the orbits of the Ehlers group in this invariant subset. The paper concludes with a spacetime characterization of the Kerr–NUT metric, which is a direct consequence of the results in this paper combined with the results in [9].

2. General identities for spacetimes with a Killing vector

In this paper, a C^n spacetime denotes a paracompact, Hausdorff, connected C^{n+1} four-dimensional manifold endowed with a C^n metric of signature $(-1, 1, 1, 1)$. All spacetimes are assumed to be oriented with metric volume form $\eta_{\alpha\beta\gamma\delta}$. $(\mathcal{M}, g)$ will denote a C^2 spacetime admitting a C^2 Killing vector field $\vec{\xi}$. The norm and twist 1-form of $\vec{\xi}$ are defined, respectively, by $\lambda = -\xi^\alpha \xi_\alpha$ and $\omega_\alpha = \eta_{\alpha\beta\gamma\delta}\xi^\beta \nabla^\gamma \xi^\delta$. In order to study spacetimes with a Killing vector of arbitrary causal character, it is useful to employ self-dual 2-forms, which are complex 2-forms $\boldsymbol{\mathcal{B}}$ satisfying $\boldsymbol{\mathcal{B}}^\star = -\mathrm{i}\boldsymbol{\mathcal{B}}$, where $\star$ is the Hodge dual operator. Our notation for p-forms is as follows. Boldface characters are used for p-forms, non-boldface characters are used for its components. For self-dual 2-forms, curly characters will be used (boldface for the 2-form and non-boldface for its components).

The 2-form $F_{\alpha\beta} \equiv \nabla_\alpha \xi_\beta$ and its self-dual associate $\mathcal{F}_{\alpha\beta} \equiv F_{\alpha\beta} + \mathrm{i}F^\star_{\alpha\beta}$ will play a fundamental role in the following. The 2-form $\boldsymbol{\mathcal{F}} \equiv \frac{1}{2}\mathcal{F}_{\alpha\beta}\,\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta$ will be called the *Killing form* throughout this paper. The Ernst 1-form $\boldsymbol{\sigma} = \sigma_\mu\,\mathrm{d}x^\mu$ associated with $\vec{\xi}$ is defined by

$$\sigma_\mu \equiv 2\xi^\alpha \mathcal{F}_{\alpha\mu} = \nabla_\mu \lambda - \mathrm{i}\omega_\mu. \tag{2}$$

Two well known (see, e.g., [16]) properties which are valid for any self-dual 2-forms $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{Y}}$ are

$$\mathcal{X}_{\mu\sigma}\mathcal{Y}_\nu{}^\sigma + \mathcal{Y}_{\mu\sigma}\mathcal{X}_\nu{}^\sigma = \tfrac{1}{2}g_{\mu\nu}\mathcal{X}_{\alpha\beta}\mathcal{Y}^{\alpha\beta}, \qquad \mathcal{X}_{\alpha\beta}Y^{\alpha\beta} = \tfrac{1}{2}\mathcal{X}_{\alpha\beta}\mathcal{Y}^{\alpha\beta} \tag{3}$$

where $\boldsymbol{Y} = \mathrm{Re}(\boldsymbol{\mathcal{Y}})$ is the real part of $\boldsymbol{\mathcal{Y}}$. We now obtain some algebraic identities for $\boldsymbol{\mathcal{F}}$. Directly from the definition of $\boldsymbol{\sigma}$ and the first equation in (3) we obtain

$$\sigma_\alpha \sigma^\alpha = -\lambda \boldsymbol{\mathcal{F}}^2, \tag{4}$$

where $\boldsymbol{\mathcal{F}}^2 \equiv \mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta}$. An important identity is

$$\eta_{\alpha\beta\mu\nu}\xi^\mu \mathcal{F}^{\rho\nu} = -\mathrm{i}\xi^\rho \mathcal{F}_{\alpha\beta} + \tfrac{1}{2}\mathrm{i}\delta^\rho_\alpha \sigma_\beta - \tfrac{1}{2}\mathrm{i}\delta^\rho_\beta \sigma_\alpha, \tag{5}$$

which can be proven by inserting $\mathcal{F}^{\rho\nu} = \frac{1}{2}\mathrm{i}\eta^{\rho\nu}{}_{\gamma\delta}\mathcal{F}^{\gamma\delta}$ into the left-hand side and expanding the products of ηs. This identity also holds for an arbitrary self-dual 2-form $\boldsymbol{\mathcal{X}}$ provided $\boldsymbol{\sigma}$ is defined accordingly (see (2)). Another identity which will be useful in section 5 is

$$\begin{aligned}\sigma_\beta \nabla_\alpha \lambda + \sigma_\alpha \nabla_\beta \lambda - g_{\alpha\beta}\sigma^\mu \nabla_\mu \lambda + 2\xi_\beta \sigma^\mu F_{\alpha\mu} + 2\xi_\alpha \sigma^\mu F_{\beta\mu} - 4\lambda F_\alpha{}^\mu \mathcal{F}_{\mu\beta} \\ = \sigma_\alpha \sigma_\beta + \boldsymbol{\mathcal{F}}^2\left(\lambda g_{\alpha\beta} + \xi_\alpha \xi_\beta\right).\end{aligned} \tag{6}$$

This expression can be proven by splitting the real and imaginary parts. The real part of (6) was already proven in [9]. The imaginary part is easily shown by using the first identity in (3) (with $\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{F}}$).

Let us consider next identities involving covariant derivatives of the Killing form and/or of the Ernst 1-form. They already involve the curvature of the spacetime. From $\nabla_\mu \nabla_\alpha \xi_\beta =$

$\xi^\nu R_{\nu\mu\alpha\beta}$, which is a well known consequence of the Killing equations, the following identity follows:

$$\nabla_\mu \mathcal{F}_{\alpha\beta} = \xi^\nu \mathcal{R}_{\nu\mu\alpha\beta}, \tag{7}$$

where $\mathcal{R}_{\nu\mu\alpha\beta}$ is the so-called *right self-dual Riemann tensor* defined by $\mathcal{R}_{\nu\mu\alpha\beta} = R_{\nu\mu\alpha\beta} + \frac{1}{2}\mathrm{i}\eta_{\alpha\beta\rho\sigma}R_{\nu\mu}{}^{\rho\sigma}$. Some well known properties of $\mathcal{R}_{\alpha\beta\gamma\delta}$ are

$$g^{\alpha\lambda}\mathcal{R}_{\alpha\beta\lambda\mu} = R_{\beta\mu}, \qquad \mathcal{R}_{\alpha\beta\lambda\mu} + \mathcal{R}_{\alpha\lambda\mu\beta} + \mathcal{R}_{\alpha\mu\beta\lambda} = \mathrm{i}\eta_{\gamma\beta\lambda\mu}R^\gamma{}_\alpha, \tag{8}$$

$$\mathcal{R}_{\alpha\beta\lambda\mu} - \mathcal{R}_{\lambda\mu\alpha\beta} = \mathrm{i}\left(\eta_{\lambda\mu\alpha\sigma}R^\sigma{}_\beta - \eta_{\lambda\mu\beta\sigma}R^\sigma{}_\alpha - \tfrac{1}{2}R\eta_{\lambda\mu\alpha\beta}\right), \tag{9}$$

$$\nabla^\alpha \mathcal{R}_{\nu\alpha\beta\mu} = \nabla_\mu R_{\nu\beta} - \nabla_\beta R_{\nu\mu} + \mathrm{i}\eta_{\beta\mu\rho\sigma}\nabla^\sigma R^\rho{}_\nu, \tag{10}$$

where $R_{\alpha\beta} \equiv R^\mu{}_{\alpha\mu\beta}$ is the Ricci tensor and R is the scalar of curvature. From the definition of the Ernst 1-form and (7) we easily find

$$\nabla_\alpha \sigma_\beta - 2\nabla_\alpha \xi^\mu \mathcal{F}_{\mu\beta} = 2\xi^\mu \xi^\nu \mathcal{R}_{\nu\alpha\mu\beta}, \tag{11}$$

which will be of fundamental importance in section 5. We now obtain identities for the divergence and the exterior derivative of the Ernst 1-form. Using (11) and of the properties of $\mathcal{R}_{\alpha\beta\gamma\delta}$, we obtain

$$\nabla_\alpha \sigma^\alpha = -\boldsymbol{\mathcal{F}}^2 + 2\xi^\alpha\xi^\beta R_{\alpha\beta}, \tag{12}$$

$$\nabla_\alpha \sigma_\beta - \nabla_\beta \sigma_\alpha = 2\mathrm{i}\xi^\nu \eta_{\nu\beta\mu\alpha} R^\mu{}_\rho \xi^\rho. \tag{13}$$

Similarly, identities for the exterior derivative and the divergence of the Killing form $\boldsymbol{\mathcal{F}}$ can be obtained directly from (8) and the fundamental equation (7). The results are

$$\nabla_\mu \mathcal{F}_{\alpha\beta} + \nabla_\alpha \mathcal{F}_{\beta\mu} + \nabla_\beta \mathcal{F}_{\mu\alpha} = \mathrm{i}\eta_{\gamma\mu\alpha\beta}\xi^\nu R^\gamma{}_\nu, \qquad \nabla_\mu \mathcal{F}^\mu{}_\beta = -\xi^\nu R_{\nu\beta}. \tag{14}$$

These expressions prove the following well known lemma, which will be needed below.

Lemma 1. *Let $(\mathcal{M}, g)$ be a smooth spacetime admitting a Killing vector $\vec{\xi}$ and let $\boldsymbol{\mathcal{F}}$ be its associated Killing form. Then, the necessary and sufficient condition for $\boldsymbol{\mathcal{F}}$ to be closed is that $R_{\alpha\beta}\xi^\beta = 0$.*

Finally, we write down identities for the covariant Laplacian of $\mathcal{F}_{\mu\nu}$ and σ_μ, i.e. $\nabla_\alpha\nabla^\alpha\mathcal{F}_{\mu\nu}$ and $\nabla_\alpha\nabla^\alpha\sigma_\mu$. Since expanding the second covariant derivatives in these expressions would lead to a rather long calculation, we recall the well known Weitzenböck formula (see, e.g., [17]) which relates the covariant Laplacian and the Hodge–Laplace operator $\boldsymbol{\Delta} \equiv \boldsymbol{d\delta} + \boldsymbol{\delta d}$ acting on p-forms, where $\boldsymbol{d}$ is the exterior differential and $\boldsymbol{\delta}$ is the codifferential, $\boldsymbol{\delta} = (-1)^p \star^{-1} \boldsymbol{d}\star$. For any p-form $\boldsymbol{\Omega}$ we have

$$(\boldsymbol{\Delta\Omega})_{\alpha_1\cdots\alpha_p} = -\nabla^\mu\nabla_\mu\Omega_{\alpha_1\cdots\alpha_p} - \sum_{q=1}^{p}(-1)^q R^\beta{}_{\alpha_q}\Omega_{\beta\alpha_1\cdots\hat{\alpha}_q\cdots\alpha_p}$$
$$+2\sum_{r<q}(-1)^{r+q}R^\beta{}_{\alpha_r}{}^\gamma{}_{\alpha_q}\Omega_{\beta\gamma\alpha_1\cdots\hat{\alpha}_r\cdots\hat{\alpha}_q\cdots\alpha_p}. \tag{15}$$

$\boldsymbol{\Delta\mathcal{F}}$ is easily calculated from (14) after using the general equation $(\boldsymbol{\delta\Omega})_{\alpha_2\cdots\alpha_p} = -\nabla^\mu\Omega_{\mu\alpha_2\cdots\alpha_p}$. The result is

$$(\boldsymbol{\Delta\mathcal{F}})_{\alpha\beta} = \nabla_\alpha\left(\xi^\nu R_{\nu\beta}\right) - \nabla_\beta\left(\xi^\nu R_{\nu\alpha}\right) - \mathrm{i}\eta_{\alpha\beta\rho\sigma}\nabla^\sigma\left(\xi^\nu R^\rho{}_\nu\right).$$

With this expression at hand, the covariant Laplacian of $\mathcal{F}_{\alpha\beta}$ follows from (15). The result takes a simpler form if the Riemann tensor is decomposed into the Ricci and the Weyl tensors

$C_{\alpha\beta\gamma\delta}$. In particular, after defining the *self-dual Weyl tensor* as $\mathcal{C}_{\nu\mu\alpha\beta} = C_{\nu\mu\alpha\beta} + \frac{1}{2}\mathrm{i}\eta_{\alpha\beta\rho\sigma}C_{\nu\mu}{}^{\rho\sigma}$, the identity becomes

$$\nabla_\alpha \nabla^\alpha \mathcal{F}_{\mu\nu} = -\tfrac{1}{2}\mathcal{F}^{\alpha\beta}\mathcal{C}_{\alpha\beta\mu\nu} + \nabla_\nu\left(\xi^\delta R_{\delta\mu}\right) - \nabla_\mu\left(\xi^\delta R_{\delta\nu}\right) + \mathrm{i}\eta_{\mu\nu\rho\sigma}\nabla^\sigma\left(\xi^\delta R_\delta{}^\rho\right) + \tfrac{1}{3}R\mathcal{F}_{\mu\nu}. \quad (16)$$

Finally, we evaluate $\nabla_\alpha\nabla^\alpha\sigma_\mu$. In this case the calculation cannot be simplified by evaluating $\boldsymbol{\Delta\sigma}$ first because we do not have an identity for $\nabla_\alpha\boldsymbol{\mathcal{F}}^2$ yet (see (12)). So, we use the definition $\sigma_\alpha = 2\xi^\beta\mathcal{F}_{\beta\alpha}$ and expand the derivatives explicitly. A somewhat long calculation using (16) and the Bianchi identity in (8) gives

$$\begin{aligned}\nabla_\alpha\nabla^\alpha\sigma_\mu = {}& -2\xi^\delta\mathcal{F}^{\alpha\beta}\mathcal{C}_{\alpha\beta\delta\mu} + \tfrac{2}{3}R\sigma_\mu - 4\xi^\delta R^\beta{}_\delta\mathcal{F}_{\beta\mu} - \sigma_\beta R^\beta{}_\mu + 2\nabla_\mu\left(\xi^\rho\xi^\sigma R_{\rho\sigma}\right)\\ &+2\mathrm{i}\eta_{\gamma\mu\alpha\beta}\nabla^\alpha\left(\xi^\beta\xi^\sigma R^\gamma{}_\sigma\right).\end{aligned}$$

Combining this expression with the Weitzenböck formula (15), the following identity for the gradient of $\boldsymbol{\mathcal{F}}^2$ is obtained

$$\nabla_\mu\boldsymbol{\mathcal{F}}^2 = 2\xi^\nu\mathcal{F}^{\alpha\beta}\mathcal{C}_{\alpha\beta\nu\mu} - \tfrac{2}{3}R\sigma_\mu + 4\xi^\nu R^\beta{}_\nu\mathcal{F}_{\beta\mu} + 2\sigma_\beta R^\beta{}_\mu.$$

3. Generalized Ehlers group

The standard definition of the Ehlers group [1, 3] is as follows. Consider a strictly stationary[1] vacuum spacetime $(\mathcal{V}, g)$ (i.e. a vacuum spacetime admitting a Killing vector which is timelike everywhere). Take the quotient set $\mathcal{V}/\vec{\xi}$ with respect to the orbits of the Killing vector. This is locally a manifold (i.e. given any point $p \in \mathcal{V}$, there exists an open neighbourhood U_p of p such that $\mathcal{N}_p \equiv U_p/\vec{\xi}$ is a manifold). There exists a well known one-to-one correspondence between tensor fields in $\mathcal{N}_p$ and tensor fields in U_p which are Lie-constant along $\vec{\xi}$ and which are completely orthogonal to $\vec{\xi}$. The symmetric tensor $\lambda g_{\alpha\beta}+\xi_\alpha\xi_\beta$ has these properties and therefore defines a symmetric tensor γ_{ij} in $\mathcal{N}_p$ which endows this space with a Riemannian structure. The Ernst 1-form σ_μ associated with $\vec{\xi}$ is closed (see (13)) and hence exact in a suitably chosen U_p. Let σ be a complex scalar in U_p satisfying $\sigma_\mu = \nabla_\mu\sigma$. This function σ defines a complex scalar in the quotient $\mathcal{N}_p$. As shown by Geroch [1], the knowledge of $\mathcal{N}_p$, γ_{ij} and σ is sufficient to reconstruct locally the original spacetime $\mathcal{V}$. The action of the Ehlers group is defined by leaving γ_{ij} invariant and transforming σ according to the Möbius map $\sigma' = (\alpha\sigma + \mathrm{i}\beta)/(\mathrm{i}\gamma\sigma+\delta)$, where $\alpha, \beta, \gamma, \delta$ are arbitrary real constants satisfying $\alpha\delta+\beta\gamma = 1$. As shown in [1], the transformed spacetime is also a solution of the Einstein vacuum field equations. The group structure of the Ehlers group can be obtained by first applying a transformation $\sigma' = (\alpha\sigma + \mathrm{i}\beta)/(\mathrm{i}\gamma\sigma + \delta)$, and then a second transformation $\sigma'' = (\alpha'\sigma' + \mathrm{i}\beta')/(\mathrm{i}\gamma'\sigma' + \delta')$ (where $\alpha'\delta' + \beta'\gamma' = 1$). The result is $\sigma'' = (\alpha''\sigma + \mathrm{i}\beta'')/(\mathrm{i}\gamma''\sigma + \delta'')$ with $\alpha'', \beta'', \gamma'', \delta''$ given by

$$\begin{pmatrix} \alpha'' & \mathrm{i}\beta'' \\ \mathrm{i}\gamma'' & \delta'' \end{pmatrix} = \begin{pmatrix} \alpha' & \mathrm{i}\beta' \\ \mathrm{i}\gamma' & \delta' \end{pmatrix}\begin{pmatrix} \alpha & \mathrm{i}\beta \\ \mathrm{i}\gamma & \delta \end{pmatrix}. \quad (17)$$

This expression shows, in particular, that the Ehlers group is isomorphic to $SL(2, \mathbb{R})$.

As discussed in the introduction, it is desirable to have a description of the Ehlers group solely in terms of spacetime objects, i.e. without passing through the manifold of trajectories. We shall start by defining a much larger group of transformations which will turn out to contain the Ehlers group as a particular case. This group of transformations is defined for an arbitrary n-dimensional manifold and it describes the fundamental underlying structure of the Ehlers group.

[1] The Ehlers group is defined similarly when the Killing vector is spacelike everywhere.

Let $\mathcal{U}$ be an arbitrary n-dimensional smooth manifold and let us denote by $\mathfrak{X}(\mathcal{U})$ the algebra of vector fields and by $\Lambda^1(\mathcal{U})$ the set of smooth 1-form fields on $\mathcal{U}$. Let us define $G \subset \mathfrak{X}(\mathcal{U}) \times \Lambda^1(\mathcal{U})$ by $G = \left\{(\vec{\zeta}, \boldsymbol{W}) \in \mathfrak{X}(\mathcal{U}) \times \Lambda^1(\mathcal{U}); \zeta^\alpha W_\alpha|_p > -1, \forall p \in \mathcal{U}\right\}$. We also define $G^{\vec{\zeta}} \equiv \{\boldsymbol{W} \in \Lambda^1(\mathcal{U}); (\vec{\zeta}, \boldsymbol{W}) \in G\}$. Let $\mathfrak{S}(\mathcal{U})$ denote the set of smooth, symmetric, two-covariant tensor fields in $\mathcal{U}$. The following map defines an action of G on $\mathfrak{S}(\mathcal{U})$

$$\begin{aligned} T: G \times \mathfrak{S}(\mathcal{U}) &\longrightarrow \mathfrak{S}(\mathcal{U}) \\ \left(\zeta^\alpha, W_\beta, g_{\alpha\beta}\right) &\longrightarrow T(\vec{\zeta}, \boldsymbol{W}, g)_{\alpha\beta} \equiv \Omega^2 g_{\alpha\beta} - \zeta_\alpha W_\beta - \zeta_\beta W_\alpha - \frac{\lambda}{\Omega^2} W_\alpha W_\beta, \end{aligned} \tag{18}$$

where $\zeta_\alpha \equiv g_{\alpha\beta}\zeta^\beta$, $\Omega^2 \equiv \zeta^\alpha W_\alpha + 1$ and $\lambda \equiv -\zeta^\alpha\zeta^\beta g_{\alpha\beta}$. Given $(\vec{\zeta}, \boldsymbol{W}) \in G$, we can also define the map

$$T^{\vec{\zeta}}_{\boldsymbol{W}}: \mathfrak{S}(\mathcal{U}) \longrightarrow \mathfrak{S}(\mathcal{U}) \tag{19}$$

$$g \longrightarrow T^{\vec{\zeta}}_{\boldsymbol{W}}(g) \equiv T(\vec{\zeta}, \boldsymbol{W}, g). \tag{20}$$

In order to show that $T^{\vec{\zeta}}_{\boldsymbol{W}}$ defines a group structure on $G^{\vec{\zeta}}$ we need to compose two such transformations. After a calculation we find

$$T^{\vec{\zeta}}_{\boldsymbol{W}^2} \circ T^{\vec{\zeta}}_{\boldsymbol{W}^1} = T^{\vec{\zeta}}_{\boldsymbol{W}^2 + \Omega_2^2 \boldsymbol{W}^1}, \qquad \text{where} \quad \Omega_2^2 = 1 + \zeta^\alpha W^2_\alpha.$$

It only remains to check that $\boldsymbol{W}^2 + \Omega_2^2 \boldsymbol{W}^1 \in G^{\vec{\zeta}}$, which follows immediately from

$$\zeta^\alpha \left(W^2_\alpha + \Omega_2^2 W^1_\alpha\right) + 1 = \Omega_1^2 \Omega_2^2 > 0, \tag{21}$$

where $\Omega_1^2 = \zeta^\alpha W^1_\alpha + 1$. Thus, the composition law in $G^{\vec{\zeta}}$ induced by $T^{\vec{\zeta}}_{\boldsymbol{W}}$ is

$$\begin{aligned} &\cdot : G^{\vec{\zeta}} \times G^{\vec{\zeta}} \longrightarrow G^{\vec{\zeta}} \\ &\left(\boldsymbol{W}^2, \boldsymbol{W}^1\right) \longrightarrow \boldsymbol{W}^2 \cdot \boldsymbol{W}^1 = \boldsymbol{W}^2 + \left(1 + \zeta^\alpha W^2_\alpha\right) \boldsymbol{W}^1. \end{aligned}$$

The unit element of $(G^{\vec{\zeta}}, \cdot)$ is obviously $\boldsymbol{W} = 0$, and the inverse of $\boldsymbol{W} \in G^{\vec{\zeta}}$ is $-\boldsymbol{W}/(1 + \zeta^\alpha W_\alpha)$. In order to investigate the group structure further, let us find a suitable set of one-parameter subgroups of $(G^{\vec{\zeta}}, \cdot)$. Let $\boldsymbol{W}^0$ be an arbitrary smooth 1-form on $\mathcal{U}$ (not necessarily satisfying $\zeta^\alpha W^0_\alpha + 1 > 0$) and let us define the smooth real function

$$f_{\boldsymbol{W}^0}(t) = \begin{cases} \dfrac{\exp(\zeta^\alpha W^0_\alpha t) - 1}{\zeta^\alpha W^0_\alpha} & \text{at points where } \zeta^\alpha W^0_\alpha \neq 0 \\ t & \text{at points where } \zeta^\alpha W^0_\alpha = 0. \end{cases}$$

Thus, $\boldsymbol{W}(t) \equiv f_{\boldsymbol{W}^0}(t)\boldsymbol{W}^0$ is a smooth 1-form in $\mathcal{U}$. It is a simple exercise to check that $\boldsymbol{W}(t)$ defines a one-parameter subgroup of $G^{\vec{\zeta}}$, i.e. that $\boldsymbol{W}(t) \in G^{\vec{\zeta}}$ and that $\boldsymbol{W}(s) \cdot \boldsymbol{W}(t) = \boldsymbol{W}(t+s)$ for all $t, s \in \mathbb{R}$. Since $\Lambda^1(\mathcal{U})$ is infinite dimensional, it follows that $(G^{\vec{\zeta}}, \cdot)$ is an infinite-dimensional group which acts on the space of symmetric 2-covariant tensors.

For definiteness, we shall now restrict ourselves to the case in which $\mathcal{U}$ is four dimensional. Our next aim is to show that $G^{\vec{\zeta}}$ leaves the subset of Lorentzian metrics on $\mathcal{U}$ invariant, i.e. that the signature is preserved under the action of $T^{\vec{\zeta}}_{\boldsymbol{W}}$.

Lemma 2. *Let $\mathcal{U}$ be a four-dimensional manifold, g be a Lorentzian metric on $\mathcal{U}$ and $\boldsymbol{W} \in G^{\vec{\zeta}}$. Then $T^{\vec{\zeta}}_{\boldsymbol{W}}(g)$ is also a Lorentzian metric in $\mathcal{U}$. Furthermore, if $(\mathcal{U}, g)$ is orientable with volume form $\eta_{\alpha\beta\mu\nu}$, then $(\mathcal{U}, T^{\vec{\zeta}}_{\boldsymbol{W}}(g))$ is also orientable with volume form $\eta'_{\alpha\beta\gamma\delta} = \Omega^2\eta_{\alpha\beta\gamma\delta}$. In addition, the inverse metric of $T^{\vec{\zeta}}_{\boldsymbol{W}}(g)_{\alpha\beta}$ reads*

$$(T^{\vec{\zeta}}_{\boldsymbol{W}}(g)^{-1})^{\alpha\beta} = \frac{1}{\Omega^2}\left(g^{\alpha\beta} + W^2\zeta^\alpha\zeta^\beta + \zeta^\alpha W^\beta + W^\alpha\zeta^\beta\right), \tag{22}$$

where $W^\alpha \equiv g^{\alpha\beta}W_\beta$, $W^2 \equiv W^\alpha W_\alpha$ and $g^{\alpha\beta}$ is the inverse of $g_{\alpha\beta}$.

Proof. We first note the following identity, valid in n dimensions:

$$\det(a_1M_{ij} + a_2b_ib_j + a_3c_ic_j) = a_1^{n-2}\det(M_{ij})\left[(a_1 + a_2b^2)(a_1 + a_3c^2) - a_2a_3(bc)^2\right], \tag{23}$$

where $a_1 \neq 0, a_2, a_3$ are arbitrary constants, M_{ij} is an arbitrary $n \times n$ invertible matrix, b_i, c_i are arbitrary n-column vectors and $b^2 = (M^{-1})^{ij}b_ib_j$, $c^2 = (M^{-1})^{ij}c_ic_j$ and $bc = (M^{-1})^{ij}b_ic_j$. This identity can be proven straightforwardly by first showing its validity when $a_3 = 0$ (which follows directly from the definition of the determinant) and then applying the result twice. We want to use (23) to evaluate the determinant of (18) in an arbitrary local coordinate system. At points $p \in \mathcal{U}$ where $\vec{\zeta}$ is non-null with respect to $g_{\alpha\beta}$ (i.e. $\lambda \neq 0$) the tensor $T^{\vec{\zeta}}_{\boldsymbol{W}}(g)$ can be rewritten as

$$T^{\vec{\zeta}}_{\boldsymbol{W}}(g)_{\alpha\beta} = \Omega^2\left[g_{\alpha\beta} + \frac{1}{\lambda}\left(\zeta_\alpha\zeta_\beta - V_\alpha V_\beta\right)\right] \tag{24}$$

where $V_\alpha \equiv \zeta_\alpha + \frac{\lambda}{\Omega^2}W_\alpha$. Applying (23) to this expression, we easily find $\det(T^{\vec{\zeta}}_{\boldsymbol{W}}(g)) = \Omega^4\det(g)$. Similarly, at those points where $\vec{\zeta}$ is null (i.e. $\lambda = 0$), $T^{\vec{\zeta}}_{\boldsymbol{W}}(g)_{\alpha\beta}$ can be rewritten as

$$T^{\vec{\zeta}}_{\boldsymbol{W}}(g)_{\alpha\beta} = \Omega^2g_{\alpha\beta} - \tfrac{1}{2}(\zeta_\alpha + W_\alpha)\left(\zeta_\beta + W_\beta\right) + \tfrac{1}{2}(\zeta_\alpha - W_\alpha)\left(\zeta_\beta - W_\beta\right).$$

Applying the identity (23) to this expression we again obtain $\det(T^{\vec{\zeta}}_{\boldsymbol{W}}(g)) = \Omega^4\det(g)$. Thus, $T^{\vec{\zeta}}_{\boldsymbol{W}}(g)$ is invertible everywhere. Expression (22) for the inverse metric can be checked by simple calculation. So, it only remains to show that the signature of $T^{\vec{\zeta}}_{\boldsymbol{W}}(g)_{\alpha\beta}$ is $(-1, 1, 1, 1)$. This is proven by noticing that any element $\boldsymbol{W} \in G^{\vec{\zeta}}$ can be continuously connected to the identity. Indeed, let $\boldsymbol{W}$ be an arbitrary 1-form satisfying $\zeta^\alpha W_\alpha + 1 > 0$ everywhere. Define $\boldsymbol{W}^0 = \boldsymbol{W}$ at points where $\zeta^\alpha W_\alpha = 0$ and $\boldsymbol{W}^0 = (\zeta^\alpha W_\alpha)^{-1}\ln\left[1 + \zeta^\beta W_\beta\right]\boldsymbol{W}$ at points where $\zeta^\alpha W_\alpha \neq 0$. It follows that $\boldsymbol{W}^0$ is a smooth 1-form and therefore we can define its associated one-parameter subgroup $\boldsymbol{W}(t)$ according to the procedure above. It is easy to check that $\boldsymbol{W}(1) = \boldsymbol{W}$. Since $\det(T^{\vec{\zeta}}_{\boldsymbol{W}(t)}(g))$ is non-zero everywhere for all t, it follows that the signature must remain unchanged. □

The group $G^{\vec{\zeta}}$ is defined for any manifold and any smooth vector field $\vec{\zeta}$. It is not even necessary that $\mathcal{U}$ be a Riemannian space. For an n-dimensional $\mathcal{U}$, the transformation defined by G contains $2n - 1$ arbitrary functions ($2n$ functions are necessary to define $\vec{\zeta}$ and $\boldsymbol{W}$ but the transformation (18) has the explicit symmetry $T(K\vec{\zeta}, K^{-1}\boldsymbol{W}, g) = T(\vec{\zeta}, \boldsymbol{W}, g)$ where K is a nowhere-vanishing scalar function). So, G defines a very general transformation. Obviously, vacuum solutions are not mapped into vacuum solutions in general. However, as we shall see below, G contains the Ehlers group as a particular case. Thus, it is plausible that there may exist other interesting subsets of G. Exploring this problem is beyond the scope of this paper and should be addressed elsewhere.

The aim of the next section is to exploit the general group structure introduced in this section in order to define the Ehlers group on spacetimes admitting a Killing vector of arbitrary causal character.

4. Spacetime description of the Ehlers group

Let us consider a spacetime $(\mathcal{M}, g)$ admitting a Killing vector $\vec{\xi}$ such that $\xi^\alpha R_{\alpha\beta} = 0$ and let us define all the objects associated with $\vec{\xi}$ as in section 2. The Ernst 1-form is closed by virtue of (13). In order to define the Ehlers group we need to impose two global requirements on $\mathcal{M}$, which are essential for the whole construction. The first one is that $\boldsymbol{\sigma}$ is exact, i.e. that there exists a complex function $\sigma \equiv \lambda - \mathrm{i}\omega$ on $\mathcal{M}$ (ω is called the twist potential), such that $\nabla_\alpha \sigma = \sigma_\alpha$. Under these circumstances, let us define the 2-form

$$4\gamma\left[\delta F^\star_{\alpha\beta} + \gamma\left(\omega F^\star_{\alpha\beta} - \lambda F_{\alpha\beta}\right)\right] = \mathrm{Re}\left[-4\gamma\left(\gamma\overline{\sigma} + \mathrm{i}\delta\right)\mathcal{F}_{\alpha\beta}\right], \tag{25}$$

where γ and δ are arbitrary, non-simultaneously vanishing, real constants (a bar denotes a complex conjugate). The following chain of equalities shows that this 2-form is closed:

$$\begin{aligned}\star\boldsymbol{d}\left(\mathrm{Re}\left[\left(\gamma\overline{\sigma} + \mathrm{i}\delta\right)\boldsymbol{\mathcal{F}}\right]\right) &= \mathrm{Re}\left[\star\boldsymbol{d}\left(\left(\gamma\overline{\sigma} + \mathrm{i}\delta\right)\boldsymbol{\mathcal{F}}\right)\right] = \gamma\,\mathrm{Re}\left[\star\left(\boldsymbol{\mathcal{F}} \wedge \overline{\boldsymbol{\sigma}}\right)\right] \\ &= \gamma\,\mathrm{Re}\left(\mathrm{i}_{\overline{\sigma}} \star \boldsymbol{\mathcal{F}}\right) = \gamma\,\mathrm{Re}\left(-\mathrm{i}\,\mathrm{i}_{\overline{\sigma}}\boldsymbol{\mathcal{F}}\right) = 0. \end{aligned} \tag{26}$$

In the second equality we have used the fact that $\boldsymbol{\mathcal{F}}$ is closed (see (14)). In the third equality we have used the general identity $\star(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) = \mathrm{i}_{\boldsymbol{\beta}} \star \boldsymbol{\alpha}$, where $\boldsymbol{\alpha}$ is any p-form, and $\boldsymbol{\beta}$ is any q-form ($q + p \leqslant \dim \mathcal{M}$) and $\mathrm{i}_{\boldsymbol{\beta}}\boldsymbol{\alpha}$ denotes interior contraction. The last equality is a consequence of $\mathcal{F}_{\nu\mu}\overline{\mathcal{F}}_\delta{}^\mu$ being symmetric and therefore real.

Now we have to impose the second global requirement on $(\mathcal{M}, g)$, namely that the 2-form (25) is exact for all values of γ and δ, i.e. there exists a 1-form $\boldsymbol{W}$ satisfying

$$\nabla_\alpha W_\beta - \nabla_\beta W_\alpha = \mathrm{Re}\left[-4\gamma\left(\gamma\overline{\sigma} + \mathrm{i}\delta\right)\mathcal{F}_{\alpha\beta}\right]. \tag{27}$$

Furthermore, we demand that there exists a solution $\boldsymbol{W}$ of (27) that satisfies

$$\Omega^2 \equiv \xi^\alpha W_\alpha + 1 = \gamma^2\lambda^2 + (\delta + \gamma\omega)^2 = (\mathrm{i}\gamma\sigma + \delta)\left(-\mathrm{i}\gamma\overline{\sigma} + \delta\right). \tag{28}$$

Since the solution of (27) is defined up to a closed 1-form, we can always achieve (28) locally. However, it is essential that (28) is also satisfied globally. In general, equations (27) and (28) do not fix a unique solution yet (we can always add a closed 1-form which is orthogonal to $\vec{\xi}$, for instance, $\boldsymbol{\sigma}$). However, provided the global conditions discussed above are fulfilled, we can associate to each pair of values γ and δ a unique 1-form $\boldsymbol{W}$ satisfying (27) and (28). We will assume from now on that such a choice has been made.

Now, we can apply the general transformation $T^{\vec{\xi}}_{\boldsymbol{W}}$ defined in section 3 with respect to the Killing vector $\vec{\xi}$ and the 1-form $\boldsymbol{W}$. In order to do that, we must ensure that $\xi^\alpha W_\alpha + 1 > 0$. Thus, the points where Ω^2 vanishes must be excluded from $\mathcal{M}$ beforehand. These points correspond to $\lambda = 0$ and $\omega = -\delta/\gamma$ (with $\gamma \neq 0$). Note that the excluded points (if any) are contained in the region where the Killing vector is null. As we shall see below, the transformed spacetime will in general contain a curvature singularity precisely at the points we have excluded (if they exist). Thus, from now on, and for each value of δ/γ, we exclude the set of points $\lambda = 0$ and $\omega = -\delta/\gamma$ from the manifold $\mathcal{M}$. In order to simplify the notation, the resulting manifold (which is in general different for each value of δ/γ) will still be denoted by $\mathcal{M}$. The meaning of $\mathcal{M}$ should become clear from the context.

We can now define the transformed metric $T^{\vec{\xi}}_{\boldsymbol{W}}(g)_{\alpha\beta}$ on $\mathcal{M}$. From section 3 we know that $T^{\vec{\xi}}_{\boldsymbol{W}}(g)_{\alpha\beta}$ is smooth and Lorentzian. Our next aim is to prove that $\vec{\xi}$ is also a Killing vector of $T_{\boldsymbol{W}}^{\vec{\xi}}(g)_{\alpha\beta}$. Then, we will prove that its Ricci tensor (which we denote by $R'_{\alpha\beta}$) satisfies $\xi^\alpha R'_{\alpha\beta} = 0$. This will allow us to compose transformations and prove that they define a group.

Lemma 3. *The vector field $\vec{\xi}$ is a Killing vector of the metric $T_{\boldsymbol{W}}^{\vec{\xi}}(g)_{\alpha\beta}$.*

Proof. By construction $\pounds_{\vec{\xi}}\,\Omega = 0$. Thus, we only need to show that $\pounds_{\xi}\boldsymbol{W} = 0$. From the definition of the Lie derivative we find $\left(\pounds_{\vec{\xi}}\boldsymbol{W}\right)_\alpha = \xi^\mu\left(\nabla_\mu W_\alpha - \nabla_\alpha W_\mu\right) + \nabla_\alpha\left(\Omega^2 - 1\right)$. Using the equation

$$\nabla_\alpha \Omega^2 = 2\gamma\,\mathrm{Re}\left[(\gamma\overline{\sigma} + \mathrm{i}\delta)\,\sigma_\alpha\right], \tag{29}$$

(which is a direct consequence of (28)) the vanishing of $\pounds_{\vec{\xi}}\boldsymbol{W}$ follows immediately. □

In order to be able to compose transformations $T_{\boldsymbol{W}}^{\vec{\xi}}$, it is necessary to determine the Killing form associated with $\vec{\xi}$ in the transformed spacetime $\left(\mathcal{M}, T_{\boldsymbol{W}}^{\vec{\xi}}(g)\right)$. This is addressed in the following lemma.

Lemma 4. *The complex 2-form*

$$\boldsymbol{\mathcal{F}}' = \frac{1}{(\delta + \mathrm{i}\gamma\sigma)^2}\left[\Omega^2\boldsymbol{\mathcal{F}} - \tfrac{1}{2}\boldsymbol{W}\wedge\boldsymbol{\sigma}\right] \tag{30}$$

is the Killing form of $\vec{\xi}$ with respect to the metric $T_{\boldsymbol{W}}^{\vec{\xi}}(g)_{\alpha\beta}$.

Proof. We must show that $\boldsymbol{\mathcal{F}}'$ is self-dual with respect to the metric $T_{\boldsymbol{W}}^{\vec{\xi}}(g)_{\alpha\beta}$ and also that $\mathrm{Re}(\boldsymbol{\mathcal{F}}') = \frac{1}{2}\boldsymbol{dV}$ where $V_\alpha \equiv T_{\boldsymbol{W}}^{\vec{\xi}}(g)_{\alpha\beta}\xi^\beta$. To prove the first part, let us raise the indices of $\boldsymbol{\mathcal{F}}'_{\alpha\beta}$ with respect to $T_{\boldsymbol{W}}^{\vec{\xi}}(g)_{\alpha\beta}$. Starting from

$$\mathcal{F}'_{\alpha\beta} = \frac{1}{(\delta + \mathrm{i}\gamma\sigma)^2}\left[\Omega^2\,\mathcal{F}_{\alpha\beta} + \tfrac{1}{2}\left(W_\beta\sigma_\alpha - W_\alpha\sigma_\beta\right)\right], \tag{31}$$

a somewhat long, although straightforward, calculation gives

$$\mathcal{F}'^{\mu\nu} \equiv \left[T_{\boldsymbol{W}}^{\vec{\xi}}(g)^{-1}\right]^{\mu\alpha}\left[T_{\boldsymbol{W}}^{\vec{\xi}}(g)^{-1}\right]^{\nu\beta}\mathcal{F}'_{\alpha\beta} = \frac{1}{\Omega^2\,(\mathrm{i}\gamma\sigma + \delta)^2}\left[\mathcal{F}^{\mu\nu} + \xi^\mu W_\alpha\mathcal{F}^{\alpha\nu} - \xi^\nu W_\alpha\mathcal{F}^{\alpha\mu}\right]. \tag{32}$$

Using identity (5) and the fact that $\eta'_{\alpha\beta\lambda\mu} = \Omega^2\eta_{\alpha\beta\lambda\mu}$ (see lemma 2), the self-duality of $\mathcal{F}'_{\alpha\beta}$ follows readily. Regarding $\mathrm{Re}(\boldsymbol{\mathcal{F}}) = \frac{1}{2}\boldsymbol{dV}$, we first recall that $V_\alpha = \xi_\alpha + \frac{\lambda}{\Omega^2}W_\alpha$. Using equations (27) and (29) together with the fact that $\nabla_\alpha\lambda = \mathrm{Re}(\sigma_\alpha)$, the equation $\nabla_\alpha V_\beta - \nabla_\beta V_\alpha = 2\,\mathrm{Re}(\mathcal{F}'_{\alpha\beta})$ follows without difficulty. □

Once we know the Killing form of $\vec{\xi}$ in the transformed spacetime, we can compute the Ernst 1-form of $\vec{\xi}$ with respect to $T_{\boldsymbol{W}}^{\vec{\xi}}(g)_{\alpha\beta}$. From the definition (2) we find

$$\sigma'_\alpha \equiv 2\xi^\beta\mathcal{F}'_{\beta\alpha} = \frac{\sigma_\alpha}{(\mathrm{i}\gamma\sigma + \delta)^2}.$$

This equation implies that σ'_α is exact on $\mathcal{M}$, i.e. $\sigma'_\alpha = \nabla_\alpha\sigma'$, where the function σ' can be written in the following form, which is valid for all possible values of γ and δ:

$$\sigma' = \frac{\alpha\sigma + \mathrm{i}\beta}{\mathrm{i}\gamma\sigma + \delta}, \tag{33}$$

where α and β are real constants satisfying $\alpha\delta + \beta\gamma = 1$.

The transformation law for $\boldsymbol{\mathcal{F}}^2$ will be required in section 6. It is found directly from (32) and reads

$$\boldsymbol{\mathcal{F}}'^2 = \frac{\boldsymbol{\mathcal{F}}^2}{(\delta + \mathrm{i}\gamma\sigma)^4}. \tag{34}$$

Let us now prove that $\xi^\alpha R'_{\alpha\beta} = 0$. This is important because it will allow us to compose transformations (recall that the only local conditions we imposed on $(\mathcal{M}, g)$ in order to define $T^{\vec{\xi}}_{\boldsymbol{W}}$ were the existence of a Killing vector $\vec{\xi}$ and that $\xi^\alpha R_{\alpha\beta} = 0$).

Lemma 5. *The Ricci tensor of* $T^{\vec{\xi}}_{\boldsymbol{W}}(g)_{\alpha\beta}$ *satisfies* $\xi^\alpha R'_{\alpha\beta} = 0$.

Proof. From lemma 1, $\xi^\alpha R'_{\alpha\beta} = 0$ is equivalent to $\boldsymbol{\mathcal{F}}'$ being closed. Since $\boldsymbol{\sigma}$ is exact and $\boldsymbol{W}$ satisfies $\boldsymbol{dW} = \mathrm{Re}\left[-4\gamma\,(\gamma\overline{\sigma} + \mathrm{i}\delta)\,\boldsymbol{\mathcal{F}}\right]$, we obtain, after making use of the definition of Ω^2,

$$\boldsymbol{d\mathcal{F}}' = \frac{-\mathrm{i}\gamma}{\delta + \mathrm{i}\gamma\sigma}\overline{\boldsymbol{\sigma}} \wedge \boldsymbol{\mathcal{F}} - \frac{\mathrm{i}\gamma\,(\delta - \mathrm{i}\gamma\overline{\sigma})}{(\delta + \mathrm{i}\gamma\sigma)^2}\boldsymbol{\sigma} \wedge \boldsymbol{\mathcal{F}} + \frac{2\gamma}{(\delta + \mathrm{i}\gamma\sigma)^2}\,\mathrm{Re}\left[(\gamma\overline{\sigma} + \mathrm{i}\delta)\,\boldsymbol{\mathcal{F}}\right] \wedge \boldsymbol{\sigma} = 0.$$

The vanishing of this expression follows from expanding the real part of the third term and using the identity $\boldsymbol{\mathcal{F}} \wedge \overline{\boldsymbol{\sigma}} + \overline{\boldsymbol{\mathcal{F}}} \wedge \boldsymbol{\sigma} = 0$, which has already been proven in (26). □

Thus, the transformations defined by $\boldsymbol{W}$ can be iterated, at least locally. However, we also had to impose two global conditions on $(\mathcal{M}, g)$ in order to define $T^{\vec{\xi}}_{\boldsymbol{W}}$. The first one (i.e. that the Ernst 1-form is exact) has already been proven in (33)). Regarding $\boldsymbol{W}$, we must ensure that the exterior system (27) in the transformed spacetime is also integrable. We address this issue as follows. Consider the transformed equation

$$\nabla_\alpha W'_\beta - \nabla_\beta W'_\alpha = \mathrm{Re}\left[-4\gamma'\left(\gamma'\,\overline{\sigma'} + \mathrm{i}\delta'\right)\mathcal{F}'_{\alpha\beta}\right], \tag{35}$$

where γ' and δ' are not simultaneously zero. The left-hand side of (35) is already known to be closed, so we only need to prove that it is also exact.

Let us assume for the moment that $\boldsymbol{W}'$ defined by (35) exists globally and let us use the results in section 3 in order to find its explicit expression. We first apply the transformation corresponding to $(\alpha, \beta, \gamma, \delta)$ to the original metric $g_{\alpha\beta}$ and construct $T^{\vec{\xi}}_{\boldsymbol{W}}(g)_{\alpha\beta}$. Then, we apply the transformation with respect to $\boldsymbol{W}'$ to obtain a second metric $T^{\vec{\xi}}_{\boldsymbol{W}'}(T^{\vec{\xi}}_{\boldsymbol{W}}(g))_{\alpha\beta}$. From the results in section 3 we know that $T^{\vec{\xi}}_{\boldsymbol{W}'}(T^{\vec{\xi}}_{\boldsymbol{W}}(g))_{\alpha\beta} = T^{\vec{\xi}}_{\boldsymbol{W}''}(g)_{\alpha\beta}$, where $\boldsymbol{W}'' = \boldsymbol{W}' + \Omega'^2\boldsymbol{W}$, and $\Omega'^2 = 1 + \xi^\alpha W'_\alpha = \left(\delta' + \mathrm{i}\gamma'\sigma'\right)\left(\delta' - \mathrm{i}\gamma'\sigma'\right)$. If it were true that the transformations defined by $\boldsymbol{W}$ form a group isomorphic to $SL(2, \mathbb{R})$, then $\boldsymbol{W}''$ should be the solution of (27) corresponding to $\alpha'', \beta'', \gamma'', \delta''$, (where these constants are given by (17)). With this information at hand, we can now prove that $\boldsymbol{W}'$ exists and has the desired properties.

Let $\boldsymbol{W}$ and $\boldsymbol{W}''$ be the unique solutions of (27) corresponding to $\alpha, \beta, \gamma, \delta$ and $\alpha'', \beta'', \gamma'', \delta''$, respectively (they exist globally on $\mathcal{M}$ by assumption). Let us define $\alpha', \beta', \gamma', \delta'$ as the solution of (17) (which is unique). We can also define σ' according to (33) and $\Omega'^2 \equiv \left(\delta' + \mathrm{i}\gamma'\sigma'\right)\left(\delta' - \mathrm{i}\gamma'\sigma'\right)$. Finally, we define

$$\boldsymbol{W}' \equiv \boldsymbol{W}'' - \Omega'^2\boldsymbol{W}. \tag{36}$$

This object exists globally on $\mathcal{M}$ by construction and we have shown that it is the only candidate for being the solution of (35) satisfying $\xi^\alpha W'_\alpha + 1 = \Omega'^2$ (i.e. the only candidate compatible with the existence of an isomorphism between the group we are constructing and $SL(2,\mathbb{R})$). We need to prove that $\boldsymbol{W}'$ thus defined actually solves the desired equations. Checking $\xi^\alpha W'_\alpha + 1 = \Omega'^2$ is easy from

$$1 + \xi^\alpha W'_\alpha = 1 + \xi^\alpha W''_\alpha - \Omega'^2 \xi^\alpha W_\alpha = \Omega'^2 + \Omega''^2 - \Omega'^2\Omega^2,$$

after taking into account the definitions of Ω, Ω'' and Ω' together with $\mathrm{i}\gamma''\sigma + \delta'' = \left(\mathrm{i}\gamma'\sigma' + \delta'\right)(\mathrm{i}\gamma\sigma + \delta)$. Finally, we need to check that the differential equation

$$\boldsymbol{dW}' = \boldsymbol{d}\big(\boldsymbol{W}'' - \Omega'^2\boldsymbol{W}\big) = \mathrm{Re}\left[-4\gamma'\left(\gamma'\overline{\sigma'} + \mathrm{i}\delta'\right)\boldsymbol{\mathcal{F}}'\right]$$

is satisfied. This can be proven by direct calculation using the differential equations satisfied by $\boldsymbol{W}''$ and $\boldsymbol{W}'$ together with the expression for $\boldsymbol{\mathcal{F}}'$ (30).

This argument proves that the composition of transformations can now be globally performed and that they form a group isomorphic to $SL(2,\mathbb{R})$. It should be remarked that $\boldsymbol{W}$ can be chosen freely for each value of $\alpha, \beta, \gamma, \delta$ *only* for the original metric $g_{\alpha\beta}$. The corresponding choice for the transformed metrics $T^{\vec{\xi}}_{\boldsymbol{W}}(g)_{\alpha\beta}$ is uniquely fixed by (36).

It is easy to check that the group we have just defined coincides with the original Ehlers group when the Killing vector is timelike (or spacelike). Indeed, this follows from the transformation law for the metric when written as in (24), and from the transformation law for the Ernst potential (33). In the approach we have followed no restriction on the causal character of the Killing vector was made. This shows that the Ehlers group exists even when the Killing vector changes its causal character throughout the spacetime. Furthermore, the global conditions necessary for the transformation to exist on a given spacetime have been clarified. This extended group of transformations will be called the *spacetime Ehlers group* in this paper.

Although the spacetime Ehlers group has been defined for smooth spacetimes admitting a Killing vector $\vec{\xi}$ which satisfies $\xi^\alpha R_{\alpha\beta} = 0$, the vacuum subcase is particularly important because the original Ehlers group maps vacuum solutions into vacuum solutions. Proving that this also holds for the spacetime Ehlers group will be our next aim. There are several methods to do this. We give a proof that exploits the group structure and which might be of interest for more general situations (i.e. for other subsets of $G^{\vec{\xi}}$).

Let us consider an arbitrary one-parameter subgroup of the spacetime Ehlers group (i.e. of $SL(2,\mathbb{R})$). The starting metric $g_{\alpha\beta}$ is transformed under this subgroup into a one-parameter family of metrics $T^{\vec{\xi}}_{\boldsymbol{W}(t)}(g)$, all of which satisfy $\xi^\alpha R_{\alpha\beta}(t) = 0$. Let us now assume that g is vacuum. The group structure of the spacetime Ehlers group implies that the one-parameter family of metrics $T^{\vec{\xi}}_{\boldsymbol{W}(t)}(g)$ is vacuum for all t if and only if the linearized Einstein equations around the metric $T^{\vec{\xi}}_{\boldsymbol{W}(t)}(g)$ are solved by the symmetric tensor $\frac{\mathrm{d}}{\mathrm{d}t}T^{\vec{\xi}}_{\boldsymbol{W}(t)}(g)$. Let us recall that the linearized Einstein vacuum field equations for a perturbation $h_{\alpha\beta}$ around a given metric $g_{\alpha\beta}$ are

$$\dot{R}_{\alpha\gamma} \equiv -\tfrac{1}{2}\nabla_\alpha\nabla_\gamma h - \tfrac{1}{2}\nabla^\beta\nabla_\beta h_{\alpha\gamma} + \tfrac{1}{2}\nabla^\beta\nabla_\gamma h_{\alpha\beta} + \tfrac{1}{2}\nabla^\beta\nabla_\alpha h_{\gamma\beta} = 0, \tag{37}$$

where $h = h_{\alpha\beta}g^{\alpha\beta}$. The following lemma is used in theorem 1 below and gives the form of the linearized equations for the case under consideration.

Lemma 6. *Let $(\mathcal{M}, g)$ be a smooth spacetime which admits a Killing vector $\vec{\xi}$ and satisfies $\xi^\alpha R_{\alpha\beta} = 0$. Let $\boldsymbol{W}^0$ be an arbitrary 1-form which is Lie-constant along $\vec{\xi}$. Then, the linearized*

Einstein field equations (37) for a symmetric tensor of the form $h_{\alpha\beta} = (\xi^\nu W^0_\nu) g_{\alpha\beta} - \xi_\alpha W^0_\beta - \xi_\beta W^0_\alpha$ *read*

$$-2\dot{R}_{\alpha\beta} = g_{\alpha\beta}\nabla_\gamma\nabla^\gamma\left(\xi^\nu W^0_\nu\right) + 2F_{\gamma\alpha}K_\beta{}^\gamma + 2F_{\gamma\beta}K_\alpha{}^\gamma + \xi_\alpha\nabla^\gamma K_{\beta\gamma} + \xi_\beta\nabla^\gamma K_{\alpha\gamma} = 0, \tag{38}$$

where $F_{\alpha\beta} = \nabla_\alpha\xi_\beta$ *and* $K_{\alpha\beta} = \nabla_\alpha W^0_\beta - \nabla_\beta W^0_\alpha$. *Furthermore, if* W^0_α *satisfies the equations*

$$\nabla_\alpha W^0_\beta - \nabla_\beta W^0_\alpha = 4a_1 F^\star_{\alpha\beta}, \qquad \xi^\alpha W^0_\alpha = 2\,(a_2 + a_1\omega), \tag{39}$$

where a_1 *and* a_2 *are arbitrary constants and* ω *is the twist potential of* $\vec{\xi}$, *then the linearized equations (38) are satisfied identically.*

Proof. Inserting the expression for $h_{\alpha\beta}$ into (37) and expanding the appropriate second covariant derivatives we obtain, after using the Killing equations for $\vec{\xi}$ and $\nabla^\mu\nabla_\alpha\xi_\mu = \xi^\mu R_{\mu\alpha} = 0$,

$$-2\dot{R}_{\alpha\beta} = 2\nabla_\alpha\xi_\gamma\nabla^\gamma W^0_\beta + 2\nabla_\beta\xi_\gamma\nabla^\gamma W^0_\alpha + \xi^\gamma\nabla_\gamma\left(\nabla_\beta W^0_\alpha + \nabla_\alpha W^0_\beta\right)$$
$$+ g_{\alpha\beta}\nabla^\gamma\nabla_\gamma\left(\xi^\nu W^0_\nu\right) + \nabla^\gamma\left[\xi_\alpha\left(\nabla_\beta W^0_\gamma - \nabla_\gamma W^0_\alpha\right) + \xi_\beta\left(\nabla_\alpha W^0_\gamma - \nabla_\gamma W^0_\alpha\right)\right]. \tag{40}$$

Transforming this expression into (38) is not difficult after using $\pounds_{\vec{\xi}}\,(\nabla_\alpha W^0_\beta) = 0$ (which is a direct consequence of $\boldsymbol{W}^0$ being Lie-constant along $\vec{\xi}$) in the third term of (40).

To prove the second part of the lemma, we first note that equation (39) is locally integrable because $\boldsymbol{F}^\star$ is closed (from $\nabla^\mu F_{\mu\nu} = \xi^\mu R_{\mu\nu} = 0$). Furthermore,

$$\nabla^\beta F^\star_{\gamma\beta} = \tfrac{1}{2}\eta_{\gamma\beta\rho\sigma}\nabla^\beta\nabla^\rho\xi^\sigma = \tfrac{1}{2}\eta_{\gamma\beta\rho\sigma}\xi^\mu R_\mu{}^{\beta\rho\sigma} = 0,$$

by virtue of the first Bianchi identities. Moreover, the imaginary part of the first identity in (3) applied to $\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{F}}$ and the imaginary part of identity (12) read

$$\nabla_\beta\nabla^\beta\omega = 2F_{\alpha\beta}F^{\star\alpha\beta}, \qquad F_{\beta\alpha}F^{\star\beta}{}_\gamma + F_{\beta\gamma}F^{\star\beta}{}_\alpha = \tfrac{1}{2}g_{\alpha\gamma}F_{\rho\sigma}F^{\star\rho\sigma}.$$

Using these expressions, the vanishing of (38) follows straightforwardly. □

We can now prove the following theorem which shows that the transformed metric of a vacuum metric is also vacuum. In this theorem, all the global conditions required for the existence of the Ehlers transformation are spelled out.

Theorem 1. *Let* $(\mathcal{M}, g)$ *be a smooth spacetime admitting a Killing vector* $\vec{\xi}$ *and satisfying the Einstein vacuum field equations. Let* $\delta, \gamma \in \mathbb{R}$ *satisfy* $\delta^2 + \gamma^2 \neq 0$. *Define* λ, $\boldsymbol{\mathcal{F}}$ *and* $\boldsymbol{\sigma}$ *as the squared norm, the Killing form and the Ernst 1-form associated with* $\vec{\xi}$. *If the two following conditions are satisfied:*

(a) the Ernst 1-form is exact, i.e. it exists a complex smooth function $\sigma \equiv \lambda - \mathrm{i}\omega$ *such that* $\boldsymbol{\sigma} = \boldsymbol{d}\sigma$;

(b) the closed 2-form $\mathrm{Re}\,(-4\gamma\,(\gamma\overline{\sigma} + \mathrm{i}\delta)\,\boldsymbol{\mathcal{F}})$ *is exact and the equation* $\boldsymbol{dW} = \mathrm{Re}\,(-4\gamma\,(\gamma\overline{\sigma} + \mathrm{i}\delta)\,\boldsymbol{\mathcal{F}})$ *admits a solution satisfying* $W_\alpha\xi^\alpha + 1 = (\mathrm{i}\gamma\sigma + \delta)\,(-\mathrm{i}\gamma\overline{\sigma} + \delta) \equiv \Omega^2$.

Then, the symmetric tensor $T^{\vec{\xi}}_{\boldsymbol{W}}(g) \equiv \Omega^2 g - \boldsymbol{\xi}\otimes\boldsymbol{W} - \boldsymbol{W}\otimes\boldsymbol{\xi} - \frac{\lambda}{\Omega^2}\boldsymbol{W}\otimes\boldsymbol{W}$ *defines a smooth vacuum metric on the spacetime* $\tilde{\mathcal{M}} = \left\{p \in \mathcal{M};\, \lambda|_p \neq 0 \text{ or } (\gamma\omega + \delta)|_p \neq 0\right\}$.

Remark. As mentioned before, conditions (a) and (b) are always fulfilled locally. So they only pose global obstructions to the existence of the Ehlers group.

Proof. From the group structure, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} T^{\vec{\xi}}_{\boldsymbol{W}(t)}(g) = \frac{\mathrm{d}}{\mathrm{d}s}\left[T^{\vec{\xi}}_{\boldsymbol{W}(s)}\left(T^{\vec{\xi}}_{\boldsymbol{W}(t)}(g)\right)\right]\Big|_{s=0}. \tag{41}$$

On the other hand, the general transformation law (18) implies that, for an arbitrary symmetric tensor $\tilde{g}_{\alpha\beta}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}s} T^{\vec{\xi}}_{\boldsymbol{W}(s)}(\tilde{g})_{\alpha\beta}\Big|_{s=0} = \left(\xi^{\mu}\left.\frac{\mathrm{d}W_{\mu}(s)}{\mathrm{d}s}\right|_{s=0}\right)\tilde{g}_{\alpha\beta} - \tilde{\xi}_{\alpha}\left.\frac{\mathrm{d}W_{\beta}(s)}{\mathrm{d}s}\right|_{s=0} - \tilde{\xi}_{\beta}\left.\frac{\mathrm{d}W_{\alpha}(s)}{\mathrm{d}s}\right|_{s=0}, \tag{42}$$

where $\tilde{\xi}_{\alpha} = \tilde{g}_{\alpha\beta}\xi^{\beta}$ and we have used the fact that $\boldsymbol{W}(0) = 0$. Combining (41) and (42) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} T^{\vec{\xi}}_{\boldsymbol{W}(t)}(g)_{\alpha\beta} = \left(\xi^{\mu}W^{0}_{\mu}\right) T^{\vec{\xi}}_{\boldsymbol{W}(t)}(g)_{\alpha\beta} - V(t)_{\alpha}W^{0}_{\beta} - V(t)_{\beta}W^{0}_{\alpha},$$

where $V(t)_{\alpha} = T^{\vec{\xi}}_{\boldsymbol{W}(t)}(g)_{\alpha\beta}\xi^{\beta}$ and $\boldsymbol{W}^{0} \equiv \frac{\mathrm{d}\boldsymbol{W}(s)}{\mathrm{d}s}|_{s=0}$. The equations satisfied by $\boldsymbol{W}^{0}$ can be obtained directly from (27) and (28) after using $\gamma(s)|_{s=0} = 0$ and $\delta(s)|_{s=0} = 1$ and read

$$\nabla_{\alpha}W^{0}_{\beta} - \nabla_{\beta}W^{0}_{\alpha} = \mathrm{Re}\left[-4\mathrm{i}a_{1}\mathcal{F}'_{\alpha\beta}\right] = 4a_{1}F'^{\star}_{\alpha\beta}, \qquad \xi^{\alpha}W^{0}_{\alpha} = 2\left(a_{2} + a_{1}w'\right),$$

where $\mathcal{F}'_{\alpha\beta}$ and ω' are the Killing form and twist potential of $\vec{\xi}$ in the metric $T^{\vec{\xi}}_{\boldsymbol{W}(t)}(g)_{\alpha\beta}$ and we have defined $a_{1} = \frac{\mathrm{d}\gamma(s)}{\mathrm{d}s}|_{s=0}$ and $a_{2} = \frac{\mathrm{d}\delta(s)}{\mathrm{d}s}|_{s=0}$. Thus, all the conditions of lemma 6 are fulfilled and we can conclude that $T^{\vec{\xi}}_{\boldsymbol{W}(t)}(g)$ satisfies Einstein vacuum field equations. The theorem follows after using that $SL(2,\mathbb{R})$ is connected and that any element of a connected Lie group G_0 can be expressed as a finite product of elements in $\exp(\mathcal{G})$ (where $\mathcal{G}$ is the Lie algebra of G_0 and exp is the exponential map) (see, e.g., [18]). □

5. Action of the Ehlers group on the Weyl tensor

Our aim in this section is to obtain an explicit expression for the Weyl tensor of the Ehlers-transformed spacetime. In order to do that we will exploit the identity (11) of section 2. More precisely, we want to evaluate the left-hand side of this identity for the transformed metric $T^{\vec{\xi}}_{\boldsymbol{W}}(g)_{\alpha\beta}$ in order to obtain an expression for the transformed Weyl tensor. It is worth pointing out that a direct calculation of the transformed Weyl tensor for the metric (18) would be quite difficult. The identities of section 2 will allow for an indirect approach to the result.

Let us start by evaluating the covariant derivative of σ'_{α} with respect to the transformed metric. To do that it is convenient to use the following identity, which is a trivial consequence of the vanishing of the torsion of a Levi-Civita connection,

$$\nabla'_{\alpha}\sigma'_{\beta} = \tfrac{1}{2}\left(\nabla_{\alpha}\sigma'_{\beta} - \nabla_{\beta}\sigma'_{\alpha}\right) + \tfrac{1}{2}\pounds_{\vec{\sigma}'}T^{\vec{\xi}}_{\boldsymbol{W}}(g)_{\alpha\beta},$$

where $\vec{\sigma}'$ is the vector obtained by raising the indices of σ'_{α} with the metric $T^{\vec{\xi}}_{\boldsymbol{W}}(g)_{\alpha\beta}$, i.e.

$$\sigma'^{\alpha} \equiv \left[T^{\vec{\xi}}_{\boldsymbol{W}}(g)^{-1}\right]^{\alpha\beta}\sigma'_{\beta} = \frac{1}{(\delta+\mathrm{i}\gamma\sigma)^{2}\,\Omega^{2}}\left[\sigma^{\beta} + \xi^{\beta}\left(W^{\mu}\sigma_{\mu}\right)\right]. \tag{43}$$

Using the fact that $\vec{\xi}$ is a Killing vector for the metric $T_{W}^{\vec{\xi}}(g)_{\alpha\beta}$, a not-too-long calculation gives

$$\nabla'_\alpha \sigma'_\beta = \nabla_\alpha \left[\frac{\sigma_\beta}{(\delta + \mathrm{i}\gamma\sigma)^2} \right] - \frac{1}{\Omega^2 (\delta + \mathrm{i}\gamma\sigma)^2} \times \left\{ \sigma_{(\alpha} \nabla_{\beta)} \Omega^2 + \sigma^\mu \left[2F_{\mu(\alpha} W_{\beta)} - 4\gamma \,\mathrm{Re}\left[(\gamma\overline{\sigma} + \mathrm{i}\delta)\, \mathcal{F}_{\mu(\alpha} \right] \left(\xi_{\beta)} + \frac{\lambda}{\Omega^2} W_{\beta)} \right) + \tfrac{1}{2} W_\alpha W_\beta \nabla_\mu \left(\frac{\lambda}{\Omega^2} \right) - \tfrac{1}{2} g_{\alpha\beta} \nabla_\mu \Omega^2 \right] \right\}, \tag{44}$$

where, as usual, round brackets enclosing indices denote symmetrization. Let us keep this expression for later use and let us now evaluate $\nabla'_\alpha \xi^\mu \mathcal{F}'_{\mu\beta}$. We start by raising one index to $\mathcal{F}'_{\alpha\beta}$ with respect to the metric $T_{W}^{\vec{\xi}}(g)_{\alpha\beta}$, i.e. we evaluate $\mathcal{F}'^{\mu}{}_{\beta} \equiv [T_{W}^{\vec{\xi}}(g)^{-1}]^{\mu\nu} \mathcal{F}'_{\nu\beta}$. The result reads

$$\mathcal{F}'^{\mu}{}_{\beta} = \frac{1}{(\mathrm{i}\gamma\sigma + \delta)^2} \left[\mathcal{F}^{\mu}{}_{\beta} + \xi^\mu W_\alpha \mathcal{F}^{\alpha}{}_{\beta} + \frac{1}{2\Omega^2} W_\beta \sigma^\mu + \frac{1}{2\Omega^2} \xi^\mu W_\beta \sigma^\rho W_\rho \right].$$

We now take into account that $\nabla'_\alpha \xi^\mu \mathcal{F}'_{\mu\beta} = \mathrm{Re}\left(\mathcal{F}'_{\alpha\mu}\right) \mathcal{F}'^{\mu}{}_{\nu}$ so that only the latter has to be calculated. A simple calculation gives

$$\mathrm{Re}\left(\mathcal{F}'_{\alpha\mu}\right) \mathcal{F}'^{\mu}{}_{\beta} = \frac{1}{(\delta + \mathrm{i}\gamma\sigma)^2} \mathrm{Re}\left[\frac{1}{(\delta + \mathrm{i}\gamma\sigma)^2} \left(\mathcal{F}_{\alpha\mu} - \frac{1}{2\Omega^2} W_\alpha \sigma_\mu \right) \right] \left[\Omega^2 \mathcal{F}^{\mu}{}_{\beta} + \tfrac{1}{2} W_\beta \sigma^\mu \right]. \tag{45}$$

To proceed further we need to use two identities. The first one is valid for any complex quantity B and reads

$$\mathrm{Re}\left(\frac{B}{(\delta + \mathrm{i}\gamma\sigma)^2} \right) = \frac{1}{\Omega^4} \left[(\delta + \mathrm{i}\gamma\sigma)^2 \,\mathrm{Re}(B) + 2B\gamma\lambda \left(\gamma \frac{\sigma - \overline{\sigma}}{2} - \mathrm{i}\delta \right) \right], \tag{46}$$

which can be proven easily from the trivial relation $\mathrm{Re}(AB) = \overline{A}\ \mathrm{Re}(B) + B(A - \overline{A})/2$. The second identity reads

$$\left(\mathcal{F}_{\alpha\mu} - \frac{1}{2\Omega^2} W_\alpha \sigma_\mu \right) \left(\mathcal{F}^{\mu}{}_{\beta} + \frac{1}{2\Omega^2} W_\beta \sigma^\mu \right) = -\frac{\boldsymbol{\mathcal{F}}^2}{4\Omega^2} T_{W}^{\vec{\xi}}(g)_{\alpha\beta}, \tag{47}$$

and it is proven by expanding the left-hand side and using standard properties of self-dual 2-forms (see section 2). Inserting (46) and (47) into (45) we obtain

$$\nabla'_\alpha \xi^\mu \mathcal{F}'_{\mu\beta} = \frac{1}{\Omega^2} \mathrm{Re}\left[\left(\mathcal{F}_{\alpha\mu} - \frac{1}{2\Omega^2} W_\alpha \sigma_\mu \right) \right] \left(\mathcal{F}^{\mu}{}_{\beta} + \frac{1}{2\Omega^2} W_\beta \sigma^\mu \right) - \frac{\boldsymbol{\mathcal{F}}^2 \gamma\lambda \left(\gamma \frac{1}{2}(\sigma - \overline{\sigma}) - \mathrm{i}\delta \right)}{2\Omega^4 (\delta + \mathrm{i}\gamma\sigma)^2} T_{W}^{\vec{\xi}}(g)_{\alpha\beta}. \tag{48}$$

We can now combine (44) and (48) in order to obtain an explicit expression for $\nabla'_\alpha \sigma'_\beta - 2\nabla'_\alpha \xi^\mu \mathcal{F}'_{\mu\beta}$. This is achieved by using $\nabla_\mu \lambda = \mathrm{Re}(\sigma_\mu)$, together with the identity $\mathrm{Re}[(\gamma\overline{\sigma} + \mathrm{i}\delta)\mathcal{F}_{\alpha\beta}] = (\gamma\sigma - \mathrm{i}\delta)\,\mathrm{Re}(\mathcal{F}_{\alpha\beta}) + \mathcal{F}_{\alpha\beta}(\mathrm{i}\delta + \gamma(\overline{\sigma} - \sigma)/2)$ (which is proven in a similar way to (47)). Two further pieces of information are required to obtain the result: identity (6) and the relation $\sigma^\mu F_{\beta\mu} + \nabla^\mu \lambda \mathcal{F}_{\mu\beta} = 0$, which can be deduced from the last equality in (26).

The calculation is now rather involved. However, many cancellations happen along the way and the final result turns out to be surprisingly simple. It reads

$$\nabla'_\alpha \sigma'_\beta - 2\nabla'_\alpha \xi^\mu \mathcal{F}'_{\mu\beta} = \frac{1}{(\delta + \mathrm{i}\gamma\sigma)^2} \left(\nabla_\alpha \sigma_\beta - 2\nabla_\alpha \xi^\mu \mathcal{F}_{\mu\beta} \right)$$

$$- \frac{3\mathrm{i}\gamma}{(\delta + \mathrm{i}\gamma\sigma)^3} \left[\sigma_\alpha \sigma_\beta + \tfrac{1}{3} \boldsymbol{\mathcal{F}}^2 \left(\lambda g_{\alpha\beta} + \xi_\alpha \xi_\beta \right) \right]. \tag{49}$$

This expression is valid for any spacetime for which the spacetime Ehlers group can be defined. No restriction to vacuum spacetimes is necessary. Let us now restrict ourselves to the vacuum case and let us define the symmetric and trace-free tensor $\mathcal{Y}_{\alpha\beta} \equiv 2\xi^\mu \xi^\nu \mathcal{C}_{\mu\alpha\nu\beta}$. Since, in vacuum, $\mathcal{R}_{\alpha\beta\gamma\delta} = \mathcal{C}_{\alpha\beta\gamma\delta}$ the transformation law of $\mathcal{Y}_{\alpha\beta}$ can be obtained directly from (11) and (49) to be

$$\mathcal{Y}'_{\alpha\beta} = \frac{\mathcal{Y}_{\alpha\beta}}{(\delta + \mathrm{i}\gamma\sigma)^2} - \frac{3\mathrm{i}\gamma}{(\delta + \mathrm{i}\gamma\sigma)^3} \left[\sigma_\alpha \sigma_\beta + \tfrac{1}{3} \boldsymbol{\mathcal{F}}^2 \left(\lambda g_{\alpha\beta} + \xi_\alpha \xi_\beta \right) \right]. \tag{50}$$

This formula is the key expression for obtaining the transformation law for the full Weyl tensor. To do that, we first rewrite (50) as follows:

$$\xi^\mu \xi^\nu \mathcal{C}'_{\mu\alpha\nu\beta} = \frac{1}{(\delta + \mathrm{i}\gamma\sigma)^2} \xi^\mu \xi^\nu \left[\mathcal{C}_{\mu\alpha\nu\beta} - \frac{6\mathrm{i}\gamma}{\delta + \mathrm{i}\gamma\sigma} \left(\mathcal{F}_{\mu\alpha} \mathcal{F}_{\nu\beta} - \tfrac{1}{3} \boldsymbol{\mathcal{F}}^2 \boldsymbol{\mathcal{I}}_{\mu\alpha\nu\beta} \right) \right], \tag{51}$$

where $\boldsymbol{\mathcal{I}}_{\mu\alpha\nu\beta} \equiv (g_{\mu\nu}\, g_{\alpha\beta} - g_{\mu\beta}\, g_{\alpha\nu} + \mathrm{i}\, \eta_{\mu\alpha\nu\beta})/4$ is the canonical metric in the space of self-dual 2-forms. The term is parentheses in (51) is self-dual in the metric $g_{\alpha\beta}$, with respect to each pair of antisymmetric indices. On the other hand, $\mathcal{C}'_{\mu\alpha\nu\beta}$ is self-dual with respect to the metric $T^{\vec{\xi}}_{\boldsymbol{W}}(g)_{\alpha\beta}$. So, if we knew a method to transform self-dual 2-forms in $g_{\alpha\beta}$ into self-dual 2-forms in $T^{\vec{\xi}}_{\boldsymbol{W}}(g)_{\alpha\beta}$, we could obtain the full transformation law for the Weyl tensor. This is addressed in the following lemma which is proven by a straightforward, if somewhat long, calculation.

Lemma 7. *Let $P^{\mu\nu}_{\alpha\beta}$ be defined as $P^{\mu\nu}_{\alpha\beta} = \Omega^2 \delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\alpha \xi^\nu W_\beta - \xi^\mu W_\alpha \delta^\nu_\beta$. Then, a 2-form $\mathcal{X}_{\alpha\beta}$ is self-dual in $(\mathcal{M}, g)$ if and only if $\mathcal{X}'_{\alpha\beta} \equiv P^{\mu\nu}_{\alpha\beta} \mathcal{X}_{\mu\nu}$ is self-dual in $(\mathcal{M}, T^{\vec{\xi}}_{\boldsymbol{W}}(g))$.*

Remark. This lemma is true for any spacetime $(\mathcal{M}, g)$ and for any pair $(\vec{\xi}, \boldsymbol{W}) \in G$, not only for the spacetime Ehlers group we are considering in this section.

Thus, let us define the tensor

$$\mathcal{B}_{\alpha\beta\gamma\delta} = \frac{1}{(\delta + \mathrm{i}\gamma\sigma)^2} P^{\mu\nu}_{\alpha\beta} P^{\rho\sigma}_{\gamma\delta} \left[\mathcal{C}_{\mu\nu\rho\sigma} - \frac{6\mathrm{i}\gamma}{\delta + \mathrm{i}\gamma\sigma} \left(\mathcal{F}_{\alpha\beta} \mathcal{F}_{\rho\sigma} - \tfrac{1}{3} \boldsymbol{\mathcal{F}}^2 \boldsymbol{\mathcal{I}}_{\alpha\beta\rho\sigma} \right) \right],$$

which is, by construction, self-dual (for each pair of antisymmetric indices) with respect to the metric $T^{\vec{\xi}}_{\boldsymbol{W}}(g)_{\alpha\beta}$. Furthermore, from $\xi^\alpha P^{\mu\nu}_{\alpha\beta} = \xi^\mu \left(\delta^\nu_\beta - W_\beta \xi^\nu \right)$, we find $\xi^\mu \xi^\nu \mathcal{C}'_{\mu\alpha\nu\beta} = \xi^\mu \xi^\nu \mathcal{B}_{\mu\alpha\nu\beta}$. It is well know, and an easy consequence of (5), that, for an arbitrary self-dual 2-form $\boldsymbol{\mathcal{X}}$ the following equation holds:

$$2\lambda \mathcal{X}_{\alpha\beta} = \left(\xi_\beta X_\alpha - \xi_\alpha X_\beta - \mathrm{i}\eta_{\alpha\beta\rho\sigma} \xi^\rho X^\sigma \right),$$

where $X_\beta = 2\xi^\alpha \mathcal{X}_{\alpha\beta}$. A similar expression exists for objects with several pairs of antisymmetric indices. Thus, we can conclude $\lambda(\mathcal{C}'_{\mu\alpha\nu\beta} - \mathcal{B}_{\mu\alpha\nu\beta}) = 0$. Now using that $\mathcal{B}_{\mu\alpha\nu\beta}$ is continuous everywhere (including the points where $\lambda = 0$), we can conclude that, for spacetimes where the Killing vector is null at most on a set with empty interior, $\mathcal{C}'_{\mu\alpha\nu\beta} = \mathcal{B}_{\mu\alpha\nu\beta}$. Summarizing, we have proven the following:

Proposition 1. *Let $(\mathcal{M}, g)$ be a spacetime satisfying the hypotheses of theorem 1. Assume further that the set of points where the Killing vector is null has an empty interior. Then, the Weyl tensor of the spacetime $(\tilde{M}, T_{\boldsymbol{W}}^{\vec{\xi}}(g))$ reads*

$$\mathcal{C}'_{\alpha\beta\gamma\delta} = \frac{1}{(\delta + \mathrm{i}\gamma\sigma)^2} P_{\alpha\beta}^{\mu\nu} P_{\gamma\delta}^{\rho\sigma} \left[\mathcal{C}_{\mu\nu\rho\sigma} - \frac{6\mathrm{i}\gamma}{\delta + \mathrm{i}\gamma\sigma} \left(\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma} - \tfrac{1}{3}\boldsymbol{\mathcal{F}}^2 \boldsymbol{\mathcal{I}}_{\mu\nu\rho\sigma} \right) \right], \tag{52}$$

where $P_{\alpha\beta}^{\mu\nu}$ is defined in lemma 7 above.

Remark. The condition that $\vec{\xi}$ is non-null almost everywhere can be shown to be unnecessary, i.e. that proposition 1 *also* holds for Killing vectors with an arbitrary causal character. The sketch of the proof is as follows. First, obtain the transformation law for the Weyl tensor under the linearized Ehlers group for an arbitrary Killing vector. The solution turns out to be the linearized version of (52). Then, the result follows by exploiting the group structure of the spacetime Ehlers group.

The transformation law (52) for the Weyl tensor is very simple indeed. In addition to the necessary factors which transform self-dual objects in g into self-dual objects in $T_{\boldsymbol{W}}^{\vec{\xi}}(g)$, the essential part of the transformation is, besides a global conformal factor $(\delta + \mathrm{i}\gamma\sigma)^{-2}$, adding to the original self-dual Weyl tensor a term proportional to

$$\left(\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma} - \tfrac{1}{3}\boldsymbol{\mathcal{F}}^2 \boldsymbol{\mathcal{I}}_{\mu\nu\rho\sigma} \right).$$

This tensor is the simplest self-dual, symmetric and trace-free object that can be constructed out of the Killing form. Taking into account that the calculations leading to this proposition are quite long, the result is surprisingly simple and elegant. Since the curvature singularities of $(\mathcal{M}, T_{\boldsymbol{W}}^{\vec{\xi}}(g))$ must be singularities in $\mathcal{C}'_{\mu\alpha\nu\beta}$, expression (52) shows that, in general, curvature singularities in $(\mathcal{M}, T_{\boldsymbol{W}}^{\vec{\xi}}(g))$ appear at the points where $\delta + \mathrm{i}\gamma\sigma = 0$ (i.e. where $\lambda = 0$ and $\omega = -\delta/\gamma$). These points were precisely those that had to be excluded from $\mathcal{M}$ in order to define the spacetime Ehlers transformation. No further singularities may appear. Thus, we have good control on the behaviour of the transformed spacetime without having to perform the Ehlers transformation explicitly, which may be a difficult task.

6. A local characterization of the Kerr–NUT metric

The transformation law (52) for the Weyl tensor allows us to select privileged subsets of solutions of the Einstein vacuum field equations, namely those that remain invariant under the Ehlers group. The transformation law given in proposition 1 suggests one of these invariant subsets. Let us consider those vacuum spacetimes admitting a Killing vector field such that

$$\mathcal{C}_{\alpha\beta\gamma\delta} = Q \left(\mathcal{F}_{\alpha\beta}\mathcal{F}_{\gamma\delta} - \tfrac{1}{3}\boldsymbol{\mathcal{I}}_{\alpha\beta\gamma\delta}\boldsymbol{\mathcal{F}}^2 \right) \tag{53}$$

holds for some complex function Q. From the results of the previous section, it is easy to see that the Ehlers transformed self-dual Weyl tensor $\mathcal{C}'_{\alpha\beta\gamma\delta}$ retains the same form (with primed quantities). Actually, this is the simplest possible invariant subset of the vacuum Einstein field equations. It is remarkable that condition (53) also appears in a completely different context, namely in a local characterization of the Kerr metric obtained in [9]. In that paper the following result was proven:

Theorem 2. *Let $(\mathcal{V}, g)$ be a smooth, vacuum spacetime admitting a Killing vector $\vec{\xi}$. Assume that $(\mathcal{V}, g)$ is not locally flat and that the following two conditions hold:*

(a) there exists at least one point $p \in \mathcal{V}$ such that $\mathcal{F}^2|_p \neq 0$;
(b) the self-dual Weyl tensor and the Killing form associated with $\vec{\xi}$ satisfy (53).

Then, the Ernst 1-form σ_μ is exact ($\sigma_\mu = \nabla_\mu \sigma$) and Q and $\mathcal{F}^2$ must take the form $Q = -6/(c-\sigma)$, $\mathcal{F}^2 = A(c-\sigma)^4$, where $A \neq 0$ and c are complex constants.

If, in addition, $\mathrm{Re}(c) > 0$ *and A is real, then the spacetime $(\mathcal{V}, g)$ is locally isometric to a Kerr spacetime.*

This theorem was not stated explicitly in this form in [9]. However, it is not difficult to see that the proof of the main theorem in that paper also proves this theorem (see [10] for a discussion).

This theorem suggests a natural question, namely which spacetimes correspond to the other values of A and c? In this section we will answer this question and will obtain a local characterization of the Kerr–NUT metric by combining the theorem above and the action of the Ehlers group discussed in this paper. The spacetimes satisfying the hypotheses of theorem 2 can be classified by the complex constants $A \neq 0$ and c. Since, in addition, the whole family is invariant under the Ehlers group (the fact that the Ehlers-transformed $\mathcal{F}'^2$ remains non-zero somewhere follows from the transformation law (34)) we can consider the action of the Ehlers group on the parameter space defined by A and c. Given that both the hypotheses and the conclusions of theorem 2 are local, all the considerations below will also be local and there are no obstructions to define and apply Ehlers transformations.

Using the transformation law for the Weyl tensor (52) and the transformation law for $\mathcal{F}^2$ (34) we easily find

$$c' = \frac{\alpha c + \mathrm{i}\beta}{\delta + \mathrm{i}\gamma c}, \qquad A' = A\,(\delta + \mathrm{i}\gamma c)^4\,, \tag{54}$$

where c' and A' are the corresponding values for the Ehlers-transformed spacetime. The parameter space defined by A and c is four dimensional and the Ehlers group has three real parameters. Thus, there must be one real function of A and c which is invariant under Ehlers transformations. It is easy to check that $A\overline{A}\,(c+\overline{c})^4$ fulfils these requirements. In order to classify the spacetimes satisfying the hypotheses of theorem 2, we need to find a unique representative of each orbit of the Ehlers group in the space (A, c). In order to do that, we note that $c + \overline{c}$ transforms as

$$c' + \overline{c'} = \frac{c + \overline{c}}{(\delta + \mathrm{i}\gamma c)\,(\delta - \mathrm{i}\gamma \overline{c})}.$$

Thus, $c + \overline{c}$ cannot change sign under an Ehlers transformation (and it must remain zero if it was originally zero). An easy inspection of the transformation law (54) shows that the orbits corresponding to $c + \overline{c} = 0$ are uniquely characterized by the unit complex number $A/|A|$ (where the vertical bars denote the norm of the complex number). More precisely, any pair $c = \mathrm{i}s_1$, $A = A_0 \exp(\mathrm{i}B)$ (with $s_1, A_0 > 0$ and B being real) can be transformed into $c' = 0$ and $A' = \exp(\mathrm{i}B)$.

Now take an arbitrary point (A, c) with $\mathrm{Re}(c) \neq 0$. It is easy to check that there always exists an Ehlers transformation that brings this point into the canonical form $c' = \mathrm{sign}(c + \overline{c})$ and $A' = -|A|\,\mathrm{Re}(c)^2$ (we have chosen a negative sign in A' just for convenience, a positive sign can also be achieved). So, the vacuum solutions satisfying the hypotheses of theorem 2 can be classified into three classes according to the Ehlers group as follows:

- those with $\mathrm{Re}(c) = 0$, for which the orbit is determined by $A/|A|$;
- those with $\mathrm{Re}(c) > 0$ and the orbit is determined by the real constant $-|A|\,\mathrm{Re}(c)^2$;

- those with $\mathrm{Re}(c) < 0$ and the orbit is determined by $-|A|\,\mathrm{Re}(c)^2$.

With this classification at hand, we can now use theorem 2 to obtain a purely local geometric characterization of the Kerr–NUT spacetime. It is well known that the Kerr–NUT family of spacetimes is obtained and exhausted by applying the Ehlers transformations to the Kerr spacetime. Thus, the following theorem follows by combining the classification discussed above and the results of theorem 2.

Theorem 3. *Let $(\mathcal{V}, g)$ satisfy the hypotheses of theorem 2. If* $\mathrm{Re}(c) > 0$ *then the spacetime $(\mathcal{V}, g)$ is locally isometric to a Kerr–NUT spacetime.*

This theorem extends a result by Krisch [19] who analyses the behaviour, under Ehlers transformations, of the most general strictly stationary vacuum solution with vanishing Simon tensor [20]. The relationship between the vanishing of the Simon tensor [21] and the characterization of Kerr given in theorem 2 is discussed in detail in [9].

We could still ask which spacetimes correspond to a zero or a negative value of $\mathrm{Re}(c)$. Without giving the proof, let us just mention that they belong to the vacuum subset of the Plebański limit of the rotating C metric [22]. They are analogous of the Kerr–NUT spacetime, but with the geometry of a certain quotient (defined by the stationary Killing vector and one of the principal null directions) being not a round 2-sphere but a Euclidean plane (when $\mathrm{Re}(c) = 0$) or a Poincaré plane (when $\mathrm{Re}(c) < 0$).

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