Russian Mathematical Surveys

LONDON MATHEMATICAL SOCIETY EST. 1865

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To cite this article: V Kh Salikhov 2008 *Russ. Math. Surv.* **63** 570

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On the irrationality measure of π

V. Kh. Salikhov

In 1953 Mahler [1] obtained the first estimate from below for the order of approximation of π by rational numbers: $|\pi - p/q| \geqslant q^{-30}$ for all $p, q \in \mathbb{N}$ with $q \geqslant q_0$.

This result was repeatedly improved later: Mignotte [2] replaced the exponent 30 by 20, and Chudnovsky [3] replaced it by $19.8899944\ldots$. In a series of papers Hata [4]–[6] decreased the exponent to $8.016045\ldots$. The aim of this work is to prove the following statement.

Theorem 1. *For all $p, q \in \mathbb{N}$ with $q \geqslant q_0$ the following inequality holds*:

$$\left|\pi - \frac{p}{q}\right| \geqslant q^{-\nu}, \quad \text{where } \nu = 7.6063\ldots. \tag{1}$$

This improvement of Hata's result is based on the integral

$$J \equiv \int_{4-2i}^{4+2i} R(x)\,dx,$$
$$R(x) = \frac{(x-4+2i)^{3n}(x-4-2i)^{3n}(x-5)^{3n}(x-6+2i)^{3n}(x-6-2i)^{3n}}{x^{5n+1}(10-x)^{5n+1}}, \tag{2}$$

where n is even, $n \to +\infty$, and the integration path from $4-2i$ to $4+2i$ passes through the saddle points x_1 and x_2 (which are indicated below) and the point $x_3 = 5$.

Let us give the proof of the theorem stated above. Clearly, $R(x) \in \mathbb{Q}(x)$, and the parity of n implies the equality $R(10-x) = R(x)$. Therefore, the partial fraction decomposition of $R(x)$ may be given as follows:

$$R(x) = P(x) + \sum_{j=0}^{5n}\left(\frac{a_j}{x^{j+1}} + \frac{a_j}{(10-x)^{j+1}}\right), \qquad P(x) = \sum_{\nu=1}^{5n-1} b_\nu x^{\nu-1}, \tag{3}$$

where all a_j are in $\mathbb{Q}$ and all b_ν are integers.

From (2) and (3) we have

$$\varepsilon_n' \equiv \frac{1}{i}\,J = \frac{1}{2}\,a_0\pi + r_1 + r_2 \qquad \text{with} \quad a_0, r_1, r_2 \in \mathbb{Q}. \tag{4}$$

To compute the asymptotics of ε_n' and a_0 as $n \to \infty$ we use the saddle point method applied to the integrals J and $a_0 = \dfrac{1}{2\pi i}\displaystyle\int_{|x|=1} R(x)\,dx$. First of all, we compute the denominator of the rational numbers a_0, r_1, and r_2.

Let

$$d(m) = \frac{(3n)!}{(3n-m)!\,m!}, \quad m = 0, 1, \ldots, 3n; \qquad d'(m) = \frac{(5n+m)!}{m!\,(5n)!}, \quad m \in \mathbb{Z}^+.$$

For $s = 0, 1, \ldots, 5n$, let

$$M_s = \{\overline{m} = (m_1, \ldots, m_6) \mid \text{ all } m_r \in \mathbb{Z}^+;\ m_r \leqslant 3n,\ r = 1, \ldots, 5;\ m_1 + \cdots + m_6 = 5n - s\}.$$

AMS 2000 *Mathematics Subject Classification.* Primary 11J82; Secondary 11J25, 11J72, 33C60.

DOI 10.1070/RM2008v063n03ABEH004543.

Finally, let $K = \{a+bi \mid a,\ b \in \mathbb{Z}\}$ denote the ring of Gaussian integers, and for $N \in \mathbb{Z}^+$ let $\theta_N = \{N/2\}$ be the fractional part of $N/2$. From (3) one readily gets a representation of the form

$$a_j = 5^{-n-1+j}2^{2.5n-2+1.5j-\theta_j}A_j, \qquad A_j \in \mathbb{Z}, \tag{5}$$
$$A_j = \sum_{\overline{m}\in M_j} \gamma(\overline{m})\rho(\overline{m}), \qquad \rho(\overline{m}) \in K, \quad \gamma(\overline{m}) = d(m_1)\cdots d(m_5)d'(m_6).$$

The following lemma is analogous to Lemma 2.2 in [6].

Lemma 1. *For a prime p, let $T_n = \{p > \sqrt{5n} \mid 1/3 \leqslant \{n/p\} < 2/5\}$,*

$$\Lambda_1(n) = \prod_{p\leqslant\sqrt{5n}} p^{[\log 5n/\log p]}, \qquad \Lambda_2(n) = \prod_{\sqrt{5n}<p\leqslant 5n} p, \qquad \Lambda_3(n) = \prod_{p\in T_n} p,$$

and $\Delta_n = \Lambda_1(n)\Lambda_2(n)/\Lambda_3(n)$. Then $(\Delta_n\gamma(\overline{m}))/j \in \mathbb{N}$ for all $j \in \{1,\dots,5n\}$ and $\overline{m} \in M_j \cup M_{j+1}$.

Applying the lemma and equalities (3) and (5), we arrive at the following statement.

Lemma 2. *Let $Q_n = 5^{n+1}2^{-2.5n+3}\Delta_n$. Then all of $Q_n a_0/2$, $Q_n r_1$, and $Q_n r_2$ are integers.*

We have $\lim_{n\to\infty} n^{-1}\log Q_n = 5 - \big(\psi(2/5) - \psi(1/3)\big) + \log 5 - 2.5\log 2 \equiv \chi$, where $\psi(x) = \Gamma'(x)/\Gamma(x)$.

The completion of the proof of the estimate (1) follows precisely the scheme of the proof of Theorem 1.2 in [6], and thus we restrict ourselves to some remarks only.

Let (cf. (2))

$$\frac{(x-4+2i)^3(x-4-2i)^3(x-5)^3(x-6+2i)^3(x-6-2i)^3}{x^5(10-x)^5} \equiv g(t) = \frac{(t^2+6t+25)^3t^{1.5}}{(25-t)^5},$$

where $t = (x-5)^2$. The saddle points of the function $g(t)$ are the roots of the equation

$$\frac{1.5}{t} + \frac{3(2t+6)}{t^2+6t+25} - \frac{5}{t-25} = 0, \quad \text{or equivalently,} \quad t^3 - 76.2t^2 - 305t - 375 = 0;$$

namely, $t_{1,2} = -1.933883\ldots \pm i0.971403\ldots$ and $t_3 = 80.067767\ldots$.

The points x_1 and x_2 involved in (2) have the form $x_{1,2} = 5+\sqrt{t_{1,2}}$, where $\operatorname{Re}\sqrt{t_{1,2}} < 0$, that is,

$$x_1 = 5 - 0.339310\ldots - i1.431438\ldots \quad \text{and} \quad x_2 = 5 - 0.339310\ldots + i1.431438\ldots.$$

Applying Remark 2.1 in [6] to the linear form $\varepsilon_n \equiv Q_n\varepsilon'_n$ (see (4)) we deduce that the inequality (1) holds for any

$$\nu > 1 - \frac{\chi + \log|g(t_3)|}{\chi + \log|g(t_1)|} = 7.60630852\ldots,$$

and the theorem follows.

The author expresses his gratitude to Yu. V. Nesterenko and W. Zudilin for their helpful remarks.

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Presented by A. V. Mikhalev
Accepted 24/JAN/08
Translated by W. ZUDILIN