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The atomic units, the kilogram and the other proposed changes to the SI

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Abstract
The close relationship between the SI units incorporating the proposed changes of Mills *et al* and the atomic system of units is discussed. It is shown that when the definition of the second is taken into account, a definition of the kilogram in terms of the mass of a photon of specified frequency results in a kilogram that implicitly depends on $\alpha^4 m_e$ and other dimensionless constants, where α is the fine structure constant and m_e is the electron rest mass. Thus the choice of definitions proposed by Mills *et al* implicitly yields a kilogram that depends either essentially on $\alpha^4 m_e$ or on the unified atomic mass unit, m_u. Consequently, the kilogram could be regarded as implicitly retaining its present status as an independently defined SI base unit with either definition. The α-dependence of the kilogram and many other SI quantities would, of course, be affected by future changes in the definition of the second.

1. Introduction

The Kibble moving coil method [1] of realizing the watt essentially determines the Planck constant in terms of the metre, kilogram and second. The kilogram is the least accurately realized of these three quantities and measurements [2] have already achieved an accuracy that is comparable with the stability of the prototype kilogram. Assuming consistent results can be obtained between independent apparatus, it is anticipated that with further improvements the method may ultimately be used to realize the kilogram within a few parts in 10^9—a limit largely set by short-term gravitational stabilities. Although relative mass measurements at the atomic mass level already surpass this accuracy, this accuracy may yet prove to be the practical limit to the mass measurements required in macroscopic metrology.

Mills *et al* [3,4] have proposed a revised set of practically realizable definitions of the SI units [5] that are explicitly based on fundamental constants [6]. Their system takes account of the probable use of the above method to provide a method of realizing or maintaining the kilogram. It also brings to fruition the dream of Maxwell [7], Johnstone-Stoney [8], Planck [9] and many others [10].

In their paper Mills *et al* have proposed essentially two distinct ways of redefining the kilogram:

(i) in terms of the mass of a photon of specified frequency and
(ii) in terms of the unified atomic mass unit m_u.

Since $m_u N_A = M^O$, where $M^O = 10^{-3}\,\mathrm{kg\,mol^{-1}}$, proposal (ii) may alternatively involve the Avogadro constant N_A. (The symbol ^{O} after a quantity Q indicates that it represents a standard quantity [11], thus Q^O. We may also write $m_u = (m_u/m_p)(m_p/m_e)$, where m_e and m_p are the electron and proton rest masses, respectively.) Due to present-day limitations in the knowledge of m_p/m_e, the second option would be slightly less accurately realizable than route (i).

The relation between the proposed SI and the atomic system of units is discussed in the next section.

2. The proposed SI and the atomic units

Ever since the standard group of fundamental constants was established at the beginning of the twentieth century they have been used to form 'units' in order to simplify expressions when developing a theory. The statement 'let $\hbar = m_e = c = 1$' or 'let $\hbar = m_e = e = 1$' indicates that a paper is being written in terms of either these 'natural units' or 'atomic units', respectively (these unit systems usually also set $\varepsilon_o = 1$). It is a trivial process to recover the full expressions at the end of the calculation. In addition to these constants, there is also the very important fine structure constant $\alpha(= e^2/4\pi\varepsilon_o\hbar c)$. This dimensionless quantity may be used to form other natural units in combination with h, m_e, e and c.

Following Atkinson [12], Petley [13] and Flowers and Petley [14] and Mohr and Taylor [6], we may see, at least to lowest order in α, how the fundamental constants are involved

Table 1. Comparison between the atomic units and some units of the SI unit system proposed by Mills *et al*.

Quantity	Atomic unit [6]	SI units resulting from the proposal of Mills *et al* [3,4]: kilogram definition (i): 1 kg $= hf_{\rm kg}/c^2$	kilogram definition (ii): 1 kg $= n'_{\rm kg}m_{\rm u}$
Time	$a_{\rm o}/\alpha^2 c = \hbar/(\alpha m_{\rm e}c^2)$	$t_{\rm s}^{\rm O} = n_{\rm s}^{\rm O}/f_{\rm Cs} = 1\,{\rm s} = n_{\rm s}^{\rm O}k_{\rm Cs}[h/\alpha^4 m_{\rm e}c^2]$	$n_{\rm s}^{\rm O}k_{\rm Cs}[h/\alpha^4 m_{\rm e}c^2]$
Length	Bohr radius, $(\hbar/\alpha c m_{\rm e})$	$l_{\rm m}^{\rm O} = (n_{\rm m}^{\rm O}/n_{\rm Cs}^{\rm O})(hf_{\rm Cs}/c^2) = 1\,{\rm m} = (n_{\rm m}^{\rm O}/n_{\rm Cs}^{\rm O})[h/\alpha^4 c m_{\rm e}]/k_{\rm Cs}$	$(n_{\rm m}^{\rm O}/n_{\rm Cs}^{\rm O})[h/\alpha^4 m_{\rm e}c]/k_{\rm Cs}$
Mass	$m_{\rm e}$	$m_{\rm kg}^{\rm O} = hf_{\rm kg}/c^2 = 1\,{\rm kg} = (n_{\rm kg}^{\rm O}/n_{\rm s}^{\rm O})hf_{\rm Cs}/c^2 = n_{\rm kg}^{\rm O}k_{\rm Cs}[\alpha^4 m_{\rm e}]$	$m_{\rm kg}^{\rm O} = n_{\rm kg}'^{\rm O}m_{\rm u} = 1\,{\rm kg}$
Electric current	$e\alpha^2 m_{\rm e}c^2/\hbar$	$i_{\rm A}^{\rm O} = n_{\rm A}^{\rm O}k_{\rm Cs}(em_{\rm e}c^2\alpha^4/h) = 1\,{\rm A}$	$n_{\rm A}^{\rm O}k_{\rm Cs}(em_{\rm e}c^2\alpha^4/h)$
Temperature	$\alpha^2 m_{\rm e}c^2/k$	$T_{\rm K}^{\rm O} = n_{\rm K}^{\rm O}\alpha^4 k_{\rm Cs}(m_{\rm e}c^2/k) = 1\,{\rm K}$	$n'_{\rm K}m_{\rm u}c^2/k$
Velocity	αc	$(l/t)^{\rm O} = c/(299\,792\,458) = 1\,{\rm m\,s^{-1}}$	$c/(299\,792\,458)$
Electric charge	e	$Q_{\rm C}^{\rm O} = n_{\rm C}^{\rm O}e = 1\,{\rm C}$	$n_{\rm C}^{\rm O}e$
Electric resistance	$\hbar/e^2$	$R_{\Omega}^{\rm O} = n_{\Omega}^{\rm O}h/e^2 = 1\,\Omega$	$n_{\Omega}^{\rm O}\alpha^{-4}(h/e^2)k_{\rm Cs}(m_{\rm u}/m_{\rm e})$
Magnetic flux	$\hbar/e$	$\Phi_{\rm Wb}^{\rm O} = n_{\rm Wb}^{\rm O}h/e = 1\,{\rm Wb}$	$n_{\rm Wb}^{\rm O}\alpha^{-4}(h/e)k_{\rm Cs}(m_{\rm u}/m_{\rm e})$

in many of the SI units in the revised system proposed by Mills *et al* [3, 4], beginning with the second, which will remain as at present.

2.1. The second

In addition to the SI units discussed by Mills *et al*, the existing definition of the second may also be expressed in terms of atomic constants. The second is presently defined in terms of the frequency of a specified hyperfine transition in caesium-133.

Following the notation of Maleki and Prestage [15], a general expression for the hyperfine splitting frequency interval, $f_{\rm hfi}$, of alkali atoms may be obtained as

$$\begin{aligned} f_{\rm hfi} &= \tfrac{8}{3}\,\alpha^2 g_I Z(z^2 n^{*3})(1 - {\rm d}\Delta_n/{\rm d}n)F(\alpha Z)(1-\delta)(1-\varepsilon) \\ &\quad\times (m_{\rm e}/m_{\rm p})cR_\infty, \\ &= \tfrac{4}{3}\,g_I Z(z^2 n^{*3})(1 - {\rm d}\Delta_n/{\rm d}n)F(\alpha Z)(1-\delta)(1-\varepsilon) \\ &\quad\times (m_{\rm e}/m_{\rm p})[\alpha^4 m_{\rm e}c^2/h]. \end{aligned}$$

Thus, the caesium frequency is given by

$$f_{\rm Cs} = [m_{\rm e}c^2\alpha^4/h]k_{\rm Cs},$$

where $k_{\rm Cs}$ is a dimensionless quantity involving several dimensionless constants. At present $k_{\rm Cs}$ is not known with an accuracy level comparable with that of caesium atomic clocks, but its value may be deduced by substituting the experimental values of $f_{\rm Cs}$ and $[m_{\rm e}c^2\alpha^4/h]$.

The standard quantity $t_{\rm s}^{\rm O}$ defining the second is

$$t_{\rm s}^{\rm O} = n_{\rm s}/f_{\rm Cs} = n_{\rm s}/(k_{\rm Cs}[\alpha^4 m_{\rm e}c^2/h]) = 1\,{\rm s},$$

where $n_{\rm s}$ is the numerical constant (9 192 631 770) defined in the present definition of the second. This expression may be compared with the atomic unit of time shown in table 1.

The hydrogen maser frequency, which is by far the most calculable microwave frequency of those involved in the very best 'atomic clocks', may be expressed [16, 17] as

$$\begin{aligned} \nu_{\rm H} &= \tfrac{8}{3}[\alpha^4(m_{\rm e}c^2/h)](\mu_{\rm p}/\mu_{\rm e})(\mu_{\rm p}/\mu_{\rm B})^2 \\ &\quad\times (1 + m_{\rm e}/m_{\rm p})^{-3}\{1 + a_{\rm e} + \tfrac{3}{2}\alpha^2 + \ldots\}, \end{aligned}$$

where $\nu_{\rm H}$ is the hydrogen hyperfine splitting frequency. Unfortunately, owing to inexact knowledge of the polarizability of the proton, this frequency calculation would be uncertain at the fractional part in 10^6 level of accuracy. However, as is the case for caesium and other alkali atoms, the hydrogen frequency still contains the same leading term: $[\alpha^4 m_{\rm e}c^2/h]$.

Transitions in the visible region and beyond are currently under investigation with near atomic clock accuracy. Several of these promise the achievement of approaching millihertz accuracy for atom or ion based frequency standards. This accuracy would surpass that of the best caesium atomic clocks. Unfortunately such transitions are unlikely to be exactly expressible in terms of fundamental constants. The atomic hydrogen 1S–2S ground-state transition in the vacuum ultraviolet is the most theoretically predictable of these candidates, although it is probably too broad for future primary frequency standard use. However, it is reasonable to infer that, if the definition of the second were to be changed to one in the optical region, then analogous to the expression for the 1S–2S frequency, the implicit involvement of the fine structure constant α would be reduced by a factor of α^3.

3. Comparison between the proposed SI and the atomic units

Thus far nature has not revealed a unique set of primary units, and different sciences use different systems, consequently the SI and other systems must continue to be subject to empirical decision by international agreement. In the proposed developments to the SI, a measurement of the fine structure constant determines either $\varepsilon_{\rm o}$ or $\mu_{\rm o}$ ($\varepsilon_{\rm o}\mu_{\rm o}c^2 = 1$), whereas they are constant quantities in the present SI. Since the accuracy of present-day measurements of the von Klitzing quantum Hall resistance considerably surpasses that of the calculable capacitor, or other possible electrical methods, this would be unlikely to cause a practical problem to SI users.

Scientists use other unit systems for particular purposes. There is, for example, still an important group of scientists who prefer to use the centimetre, gram, second (cgs) system of units and related systems, where either $\varepsilon_{\rm o}$ or $\mu_{\rm o}$ (where $\varepsilon_{\rm o}\mu_{\rm o}c^2 = 1$) is a constant whose value is set at unity in the corresponding cgs electrostatic or electromagnetic system, respectively. Their user requirements may have to be taken into account in contemplating changes to the SI. For example, since

$$\alpha = e^2/4\pi\varepsilon_{\rm o}\hbar c,$$

the interpretation of any significant time variation of α would differ between users of the cgs systems and their derivatives and those using the proposed new SI and the two proposed kg definitions. There is increasing overlap between cosmology, where the cgs system prevails, and investigations of time variations of α via comparisons between different types of present-day atomic clocks. If a non-zero value for $\mathrm{d}\alpha/\mathrm{d}t$ were to be conclusively found it would imply that one or more of the four dimensioned constants involved was also changing. The difference between the two unit systems might lead to apparent violations of the principle that neither the name nor the symbol for a quantity should imply any particular choice of unit [18].

The present and proposed SI units are compared with the atomic units in table 1. For convenience the quantity symbols t_s^O, l_m^O, m_kg^O, i_A^O, $\Phi_\mathrm{Wb}^\mathrm{O}$, Q_C^O, R_Ω^O, T_K^O have been introduced in this paper to represent the standard quantities defining the respective SI units. Also n_x^O, etc are the standard numbers contained in the definitions of the respective units. Thus, $n_\mathrm{s}^\mathrm{O} = 9\,192\,631\,770$, $n_\mathrm{m}^\mathrm{O} = 1/299\,792\,458$, and so on. Since $m_\mathrm{e} = (m_\mathrm{e}/m_\mathrm{u})m_\mathrm{u}$, one could express the quantities in terms of m_u rather than m_e. This would reduce the realization accuracy to $\sim$2 parts in 10^9 at present. Much depends on the extent to which macroscopic measurements of gravitational mass will progress beyond this level of accuracy in the future. For simplicity, some of the dimensionless constants (including reduced mass factors and higher order terms) have been subsumed into the numerical factors k_Cs, etc. The dimensionless term k_Cs in the expression for f_Cs is not known nearly as well as the caesium frequency that may be realized with atomic clocks.

Aside from the benefit to science brought about by removing the common correlated uncertainty that dominates the values of many fundamental constants, the proposed redefinition of the kilogram in terms of a photon mass renders it a unit that is dependent on the prior definition of the second. However, it is apparent from table 1 that it thereby makes the kilogram primarily dependent on the mass of the electron, the fine structure constant and other dimensionless quantities rather than simply dependent on the Planck constant. The proposals of Mills *et al* [3, 4] therefore correspond to a choice between a kilogram that is either basically dependent on $\alpha^4 m_\mathrm{e}$ and other dimensionless constants or on the atomic mass unit m_u. This in turn changes the expression of SI units in terms of atomic constants involving energy such as the joule, watt, kelvin and volt. As with previous changes to the definitions of the SI units [13], the actual powers of α involved in table 1 would reflect the α-dependence of any future change to the definition of the second.

Note added in proof.
If continuity of the atomic quantity involved in the definition is a consideration, it may be of interest that the present definition of the kilogram may be represented in terms of atomic constants as

$$1\,\mathrm{kg} = m_\mathrm{kg}^\mathrm{O} = \sum A_\mathrm{r} N_\mathrm{r} m_\mathrm{u} + \delta, \qquad (1)$$

where there are N_r entities of relative atomic mass A_r, and the sum is taken over the isotopes of all of the known elements (currently between 112 and 120). The additional term δ allows for the mass equivalent of the binding energy of the metal, and other effects. The dominant components of the Prototype kilogram are the isotopes of platinum and iridium.

4. Conclusion

It has been shown that, when the definition of the second is expressed in terms of fundamental constants, a kilogram defined in terms of the mass of a photon is expressible in terms of $\alpha^4 m_\mathrm{e}$. The moving coil measurement of h, in terms of the metre, kilogram and second, similarly 'measures' h in terms of a quantity that depends qualitatively on h. Consequently, the quantity thereby measured still provides a method of realizing the kilogram that is entirely consistent with either of the proposed definitions of Mills *et al*.

A more explicit correspondence between the atomic units and the proposed revised SI than hitherto will aesthetically satisfy a large number of scientists. Different fundamental physical constants play a role as reference quantities in different parts of physics and, pending the success of Grand Unification Theories, those constants dominant in high energy nuclear physics differ from those dominating quantum electrodynamics (QED). Bordé [19] has discussed their role in QED with particular regard to the Planck constant. The ultimate SI may utilize very different phenomena than the QED dominated system of today. The exponent of α involved in many of the SI units is likely to change with future advances in measurement technology, particularly when the definition of the second is changed. Time and frequency are the most accurately realizable of the SI units and provide the key to measurement accuracy in modern metrology.

It is interesting that the exact correspondence between the SI and the natural units of science must await a definition of the second that is exactly calculable. This provides a formidable challenge to future theoreticians and metrologists alike. However, the SI is a 'man-made' practical system of units and the prime metrological consideration is for the SI to provide the most accurately realizable set of reference quantities for science and technology. Hence for metrological purposes the SI would be expressed in terms of t_s^O, but care would be required when interpreting the philosophical implications for atomic constants. Thus, a kilogram defined in terms of (hf_kg/c^2) becomes $n_\mathrm{kg}^\mathrm{o}(h/c^2)/t_\mathrm{s}^\mathrm{O}$. Although this kilogram definition appears to 'fix' the Planck constant in the SI, it simply fixes h in terms of a quantity that is proportional to h. As exemplified by the definition of the metre, the definitions of future SI units are unlikely to contain explicit mention of a fundamental constant.

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