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SHORT COMMUNICATION

On the algebra of quantities and their units

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Abstract

The International System of Units is presented in the SI brochure, where not only are the base units of the System defined, but many of the System's derived units are also proposed. The latter are of quantities that the brochure names but does not define; they are well-known quantities or are derived in definitions to be found elsewhere. However, many of the brochure's derived units have been subject to algebraic manipulation that makes them unrecognizable as units of the defined quantity. They do not conform to the VIM's definition of 'unit' and are in no sense units of the quantities with which they are associated. This paper questions their usefulness and suggests that derived units be defined only by the definitions of their quantities and of the base units of the system. They cannot be simplified algebraically and remain units of those quantities.

1. Quantities

A *quantity* is defined in the International Vocabulary of Basic and General Terms in Metrology [1] (VIM, from its French title) as an 'attribute of a phenomenon, body or substance that may be distinguished qualitatively and determined quantitatively'. It cannot therefore, according to this definition, have an existence independently of a phenomenon, body or substance, and it has a magnitude.

The quantities that are the subject of this paper are typically those of the system known informally as the International System of Quantities (ISQ). Quantities of this particular System are defined by Technical Committee 12 of the International Standards Organisation [2]. Such a system comprises a small number of 'base' quantities and an indefinite number of 'derived' quantities. The former are not formally defined and their choice in any system is, to some extent, arbitrary and historically based; they comprise quantities like lengths, masses, time intervals, temperatures and electric currents. They are mostly quantities whose names we encounter in numerous contexts and thereby form a concept of what they mean, but without being able to define them formally and without circularity. From natural laws derived from observations, we can use the relationships between base quantities that these laws express to derive other quantities

that are of interest. These laws are commonly expressed as algebraic functions of base quantities in what are called the defining equations of the derived quantities. In the equations, the base quantities may be raised to positive or negative powers and are related factorially. More complex, derived quantities are expressed as functions of both base quantities and other, previously derived quantities.

The VIM's definition of quantity quoted above could have been left at that. However, it goes on to say in Notes that the word has *two* senses. The examples of the first, described as 'particular quantities', are the length of a given rod, the electrical resistance of a given specimen of wire, the amount of substance concentration of ethanol in a given sample of wine—all clearly quantities according to the definition. The examples quoted of the second sense are length, time, mass, temperature, etc, all names of 'attributes' of nothing in particular, divorced from any phenomenon, body or substance and having no magnitude. In this paper, I shall call them 'kinds of quantities', not to be confused with quantities that conform to the VIM's definition.

In another Note following the definition of *quantity*, the VIM states, 'Quantities that can be placed in order of magnitude relative to one another are called *quantities of the same kind*'. In a further note it says, with examples, that quantities of the same kind may be grouped into *categories*

of quantities. The examples are:

- work, heat, energy
- thickness, circumference, wavelength.

I use the word *kind* to mean the common characteristic of quantities of the same category or kind, other than their common dimension. I shall return to this definition later.

The definition of a derived quantity is a statement naming the base quantities or other derived quantities used in the definition, their roles in the systems of which the quantity is an attribute, and their relationships with each other. It is summarized in an algebraic equation where words, even whole clauses, are replaced by letter symbols, and the ‘equals’ sign replaces the verb. As an example, the equation

$$v = \frac{l}{t}$$

is much more than a representation of the sentence, ‘Velocity (v) is length (l) divided by time (t)’. The terms velocity, length and time without further elaboration in such a statement are not determined quantitatively and the sentence has not associated them with phenomena, bodies or substances. Used thus, without reference to any system, they are of different *kinds* of quantities and the algebraic expression has no meaning. The equation must be read as, ‘The mean velocity (its scalar part, v) over a period of time (t) of a moving point is the distance (l) that it travels along its flight path in that time, divided by the length (t) of the time period’. The sentence assumes, of course, that a quantity of one kind can indeed be multiplied or divided by another of a different kind.

A simple example of a kind of quantity in the ISQ is that called area. The area a of a rectangle of sides l_1 and l_2 is said to be $a = l_1 l_2$. The equation tells us in summary that for any particular rectangle the value of a is the result of multiplying whatever the value of l_1 is by whatever the value of l_2 may be. Such a multiplication cannot take place until l_1 and l_2 have each been assigned a value. Another way of looking at it is to recognize that the expression $l_1 l_2$ is an instruction to multiply the value of l_1 by that of l_2 to obtain their product, which then equals a .

If quantities of the same kind are quantities that can be placed in order of magnitude relative to each other, their magnitudes must be comparable. But it is not immediately obvious what criteria determine whether quantities are comparable or not. It does not appear to be sufficient that they be of the same dimension. Some of the quantities used in defining a derived quantity may be vectors, but only their scalar parts are used in determining the dimension. Thus, work and torque are not comparable though of the same dimension, and the same is true of compressive stress and shear stress. Of the dimensionless quantities, although they are all numbers that can be placed in order, few people would consider them all to be quantities of the same kind. Differently defined, dimensionless quantities are not comparable.

A base quantity in a system may appear under different names, such as a length, breadth, height, depth, surface-wave amplitude, wavelength, circumference and radius. In principle, all of these examples can be, in Einstein’s phrase, measured with a ruler, and are all of one kind of base quantity, usually called length as a collective name for their common kind.

Derived quantities need to be of the same defining equation if they are to be comparable. For example, all velocities (speeds) are defined in the same way; the same definition of the term is applied whatever the system in which the velocity may occur. However, work, internal energy, potential energy and kinetic energy are all defined in different ways, all having the same dimension. But each can be converted to any of the others, so a basis of comparison exists, and they are generally considered to be of the same kind. A rate of area coverage (with paint, for example) has the same dimension, $L^2 T^{-1}$, as a kinematic viscosity, but a wall decorator would not regard his measure of output as being comparable with a kinematic viscosity; the quantities are not of the same kind.

2. Units

The VIM defines *unit* as:

particular quantity, defined and adopted by convention, with which other quantities of the same kind are compared in order to express their magnitudes relative to that quantity.

In fact, only the base units of a coherent system of units are defined and adopted by convention. The derived units are defined as functions of those base units by the defining equations of the derived quantities of which they are units. I suggest that to be comparable they must be ‘of the same kind’; they must have the same defining equations.

The defining equation of a derived quantity has on its right-hand side an expression containing the other quantities used in the definition, which may be base quantities or other derived quantities. Under the rules of algebra some of those defining quantities may cancel, leading to an expression that is of the same dimension as the first but is no longer the defining expression.

The value of a quantity of the kinds that we are discussing can be expressed, according to the VIM, only by comparing the quantity with another of the same kind: a ‘unit’. The unit may be one of a system such as the International System of Units (SI) [3]. The value is the product of a number and the unit. Such a statement may be expressed for a value of a quantity X with the notation:

$$X = \{X\} \cdot [X], \quad (1)$$

where $\{X\}$ is a number and $[X]$ is a unit of the quantity X .

Then the product of two values of the quantities X and Y is

$$XY = \{X\} \cdot \{Y\} \cdot [XY], \quad (2)$$

where $[XY]$ is a new unit, a unit of the quantity XY , assuming that XY defines a quantity. It is instructive to consider whether $[XY] = [X] \cdot [Y]$.

More generally, we can write

$$\phi(WXY \dots) = \phi(\{W\} \cdot \{X\} \cdot \{Y\} \dots)[\phi], \quad (3)$$

where W, X, Y, \dots are quantities which may be raised to positive or negative powers and are in a factorial relationship.

If, in equation (3), W, X, Y, \dots are replaced by their units $[W], [X], [Y], \dots$ we have

$$\phi([W] \cdot [X] \cdot [Y] \dots) = [\phi]. \quad (4)$$

$\phi([W] \cdot [X] \cdot [Y] \cdot \dots)$ is a quantity of the same kind as that of $\phi(WXY \dots)$ and so may serve as the expression of its unit $[\phi]$. However, some of the quantities W, X, Y, \dots may be *different quantities of the same kind*, whereas their units are *identical* quantities of that kind and may cancel when entered in the function $\phi([W] \cdot [X] \cdot [Y] \cdot \dots)$ to give a different function $\phi'([T] \cdot [U] \cdot [V] \cdot \dots)$. Here, in ϕ' , the quantities in the square brackets are units of quantities, all of different kinds, or their powers, obtained when the exponents of like units in $\phi([W] \cdot [X] \cdot [Y] \cdot \dots)$ are resolved. ϕ' has the same dimension as ϕ , but is not of the same kind. It would not satisfy the VIM's definition of a unit of ϕ .

The mean velocity of a moving point was defined above as the ratio of two quantities of different kinds, l and t . One quantity that is a mean velocity is that of a point that moves a distance of one metre in one second. Because both the metre and the second are well-defined quantities, in fact they are respectively SI units of quantities of length and time, a velocity of one metre per second may be adopted as a unit of velocities. It is obviously, as is required, a quantity of the same kind as that of any velocity with which it may be compared.

Now consider the linear rate of volumetric fuel consumption of a road vehicle. Its mean value c over a distance l is calculated as the volume V of fuel consumed in that distance, divided by l . Its defining equation is

$$c = \frac{V}{l}. \quad (5)$$

The volume V may be replaced by l_1^3 where l_1 is the length of an edge of a cube whose volume is equal to V . Equation (5) then becomes

$$c = \frac{l_1^3}{l_2}, \quad (6)$$

where l_2 is the distance travelled. The two l s must be given different subscripts because they are specified independently. A normal, sensible unit might be cubic decimetre per kilometre, and the coherent unit in the SI is cubic metre per metre. But if metric units are fed into the defining equation *and the indices of the length units are resolved*, the first reduces to micro square metre and the second to square metre. Both are units of area, but no area has been defined for the system in question, and an area is not a quantity of the same kind as that of fuel consumption¹. Neither result is an acceptable unit of that quantity, according to the VIM's definition. A person considering the purchase of the vehicle would not understand such a unit, and indeed in some countries the law allows neither it nor its reciprocal as an expression of fuel consumption to be quoted in a car salesroom.

Every quantity that is not dimensionless requires a unit by which its magnitude may be expressed. The unit must be identified by either a recognized, previously defined name, or defined using the defining equation of the quantity, with units replacing the symbols for quantities. Those quantities are identified by *their* previously defined names. Thus, for

¹ The dimensionally equivalent area of a linear volumetric fuel consumption is equal to the cross-sectional area of a stationary thread of fuel that, if picked up and consumed as its only source of fuel by the vehicle as it travelled, would be necessary and sufficient to power the vehicle. This area is not, however, the definition of linear rate of fuel consumption, despite the interesting relationship.

example, the dynamic viscosity μ of a Newtonian fluid is defined as the shear stress s in the fluid in planar laminar flow divided by the velocity gradient v/l in the fluid perpendicular to the planes of shear:

$$\mu = \frac{s}{v/l}, \quad (7)$$

where the shear stress s is a shear force divided by the area to which it is applied and the velocity gradient is the difference in velocity v between shear planes distance l apart, divided by that distance. Taking account of the definition of a force a shear stress is equal to the acceleration a of a body subjected to that force, multiplied by the body's mass m and divided by the area l_1^2 to which the force is applied, that is

$$s = \frac{ma}{l_1^2}. \quad (8)$$

If the accelerated body starts from rest and is accelerated over the distance l_1 in time t , the acceleration is the velocity l_1/t that it acquires in time t , divided by t , or

$$a = \frac{l_1}{t^2}, \quad (9)$$

and

$$s = \frac{m}{l_1 t^2}. \quad (10)$$

Equation (10) would have to be put into words thus: 'A shear stress equals the mass that, when accelerated from rest, travels a certain distance in a certain time, and is divided by that distance and the square of that time'. Philosophers have written volumes about the meaning of the 'equals' sign, and here we have two meanings. If the definition of shear stress that follows equation (7) is written as an algebraic equation, the 'equals' sign replaces and means 'is'. In equation (10), the 'equals' sign means 'has the same dimension as'. Equation (10) is not the defining equation for shear stress.

The velocity gradient is the difference v in the velocities of two shear surfaces that are separated by the distance l_1 , divided by that distance. Or it is equal to $(l_2/t)/l_1$, where l_2/t is that difference of velocity v .

So,

$$\mu = \frac{m/(l_1 t^2)}{(l_2/t)/l_1}, \quad (11)$$

$$= \frac{m}{l_2 t}. \quad (12)$$

The defining equation for dynamic viscosity is equation (7). Equation (7) reduces algebraically to equation (12) when dimensionally equivalent functions of base quantities are substituted for the derived quantities in the defining equation. The right-hand side expression of equation (12) is of the same dimension as that of equation (7), but it does not define dynamic viscosity. A viscometer could not use equation (12) as its operating principle.

The quantity's unit is $\text{Pa}/((\text{m/s})/\text{m})$, or $(\text{N}/\text{m}^2)/((\text{m/s})/\text{m})$, or $(\text{kg m}/(\text{s}^2 \text{m}^2))/((\text{m/s})/\text{m})$. Any of those expressions could serve as a coherent unit of dynamic velocity in the SI, the first

two easily recognizable as satisfying the quantity's definition. The SI brochure, table 3, gives the unit as Pa s or as kg/(m s), the later obtained by substituting base units in equation (12).

The second unit of dynamic viscosity (kg/(m s)) of the SI brochure may be seen, with a little thought, to be that of another, recognizable quantity, but not a unit of viscosity. The mass m of a wire divided by its length l has the unit kg/m. In a wiredrawing process that quantity m/l changes by the amount $\delta(m/l)$ in time δt as a section of the wire passes through the die. The mean rate of change of linear density is $\delta(m/l)/\delta t$ and the unit is recognizably kg/(m s)—the SI brochure's unit, 'expressed in base units', for the dynamic viscosity of a fluid. It could equally be the symbol of the unit of the linear rate of loss of mass by evaporation from a wet string.

It is tacitly assumed that any number of quantities and their powers may be multiplied together and the result is always another quantity, even though it may be beyond the wit of man to imagine what that other quantity is. But whether or not that is true, such a quantity, real or imagined, is of no use as a unit for a real, known quantity that is not of the same kind.

Where ostensibly the same quantity is defined in different ways the unit may differ to reflect that difference of definition. An 'amount of rainfall' is normally expressed using millimetre as the unit. Ordinary people have no difficulty in associating this quantity with the change of depth that occurs where the rain remains where it falls, such as in a swimming pool. People with a scientific bent of mind might think of the amount of rainfall as the volume of rainwater that falls on a given area, divided by that area, in which case the appropriate coherent unit for that model would be m³/m². Reducing that expression to metre or, more conveniently, millimetre, would make the 'unit' a quantity that was not that of that definition, though happily understood by the public.

A unit is a quantity of the same kind as that of the quantities for which it serves to specify values. Then if, in the expression of a derived unit, the symbols of the defining units in the expression are replaced by the symbols of the quantities of which they are units, the result should be the defining function of the derived quantity, apart from any numerical coefficient. That rule is satisfied by relatively few of the units listed in tables 3 and 4 of the SI brochure. The unit for dynamic viscosity has already been cited as an example, but there are many others, and there are several anomalous examples, all associated with the former 'supplementary units'. A relative area is said to be m²/m², which conforms to the rule, and then that it is 1, which does not. I have argued elsewhere [4] that there is no need of units for relative quantities; such quantities are merely numbers. Some 'non-coherent' ratios can, however, be useful, such as inch/mile for the scale of a map whose natural scale is 1 : 63360.

The SI brochure assigns what it describes as the 'same' unit, J/(kg K), to both specific heat capacity, $C = dq/dT$, and specific entropy S , defined by $dS = dq/T$, where dq is a change of specific energy, dT is an associated change of temperature of the system and T is its thermodynamic temperature. Obviously, quantities of these two kinds are not alike. They do not, in fact, have the same unit, and the similarity of the names is due to the ambiguity in the name kelvin (K). It has two possible and distinct meanings. n K may mean a *difference* (or change) of temperature of n units

anywhere on the Kelvin or Celsius scales, or it may mean a temperature, a *point* on the Kelvin scale of thermodynamic temperature. When 'K' is interpreted in the former sense, the unit J/(kg K) is clearly that of a specific heat capacity, being itself a specific heat capacity—the unit J/kg is a specific energy and K is a difference of temperature. If 'K' has the second meaning, J/(kg K) is the change of entropy of a system of mass one kilogram receiving reversibly one joule of heat at the constant thermodynamic temperature 1 kelvin. The units of specific heat capacity and specific entropy are not the same, and the names and symbols look the same only because of the ambiguity of the name kelvin and its symbol.

It is worth noting that each of what the SI brochure calls the SI derived units 'expressed in terms of SI base unit[s]' is the dimension of the quantity with the SI base units substituted for the corresponding base dimensions. The concept *dimension of a derived quantity*, so difficult to define, could be defined as a function of base units, of any system of units devised for the relevant system of quantities, that is algebraically equivalent to that of the unit of the quantity in the same unit system. The symbols L, T, M, A, etc are then defined as representing base units for lengths, temperatures, masses, electrical currents, etc in any system of units for the relevant system of quantities.

3. Conclusion

If many units of the SI brochure do not conform to the VIM definition, one must question their usefulness or that of the definition. The concept 'unit' is very well established and the VIM definition conforms well to it. Like is compared with like. For two quantities to be 'alike' they must be of the same kind, defined in the same way. There is nothing gained in any metrological sense by comparing a quantity with another of a quite different kind. Little is understood, for example, by a statement that the viscosity of a certain fluid at a certain state is, say, 8.4 Pa s. If the abbreviations Pa and s are recognized as being of SI units, the statement conveys the information, to anyone in possession of the SI brochure, that the value is expressed by reference to the coherent unit of the SI for dynamic viscosity, which is identified by the expression Pa s. It is obvious to anyone who knows what a viscosity is that the *quantity* Pa s, if it is a quantity, is not itself a viscosity, so the expression is in no sense a unit of viscosity. The unit itself is Pa/((m/s)/m), the two metre units in that expression being for different quantities. One can see how the brochure's identifier is derived from the unit, but the unit cannot be derived from the identifier.

The *Conférence générale des Poids et Mesures* (CGPM) and the committees that advise it, responsible for the SI, do not define derived quantities; they define units for the base quantities of a previously defined and developing system of quantities: the ISQ. All other units of the SI are derived by the definitions of the quantities of the ISQ—or indeed of any quantity that can be derived using the base quantities and derived quantities of the ISQ. The units are examples of those quantities, using the base units and other, previously derived units and ignoring constants of definition. The SI performs a service in giving handy names to the units of certain, commonly occurring, derived quantities such as 'newton' for the unit of

force and 'pascal' for that of pressure. Anyone knowing the definition of a quantity can define its unit, and one must suppose that anyone wishing to state or use a value of a quantity knows what the quantity is. The SI brochure does not define it for him or her. As an alternative to giving the actual SI unit in full, it might be sufficient to state a value of, for example, a dynamic viscosity as n si, where n is a number and 'si' is a symbol for 'coherent unit of the International System of Units'. The symbol si could be preceded by a decimal prefix. The name or symbol of the unit should not be used to indicate the kind of quantity being evaluated. The current SI brochure gives names of derived quantities without their definitions, and gives names of or expressions for their units; one has to look

elsewhere to discover what the definitions are and, in many instances, must define for oneself the actual units as that term is defined.

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