ADDENDUM TO THE PAPER "THE PROBLEM OF STRONG APPROXIMATION AND THE KNESER-TITS CONJECTURE FOR ALGEBRAIC GROUPS"

1970 Math. USSR Izv. 4 784

(http://iopscience.iop.org/0025-5726/4/4/C05)

Izv. Akad. Nauk SSSR
Ser. Mat. Tom 34 (1970), No. 4

Math. USSR–Izvestija
Vol. 4 (1970), No. 4

ADDENDUM TO THE PAPER "THE PROBLEM OF STRONG APPROXIMATION AND THE KNESER-TITS CONJECTURE FOR ALGEBRAIC GROUPS"

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UDC 513.6

Abstract. This paper contains several additional observations which serve to revise and further unify the proof of the approximation theorem.

In the paper mentioned in the title ([1]; see also [2]) the approximation problem for algebraic groups is completely solved by a new method. We offer below several additional observations; in particular, we revise the proof of the approximation theorem for groups of type A_n.

We retain the basic notation of [1].

1. In the proof of the fundamental theorem in [1] we were forced to consider separately the case of a group G for which $G_k \cong SL(1, D)$, where D is a division ring, since our method is inapplicable because of the fact that for certain p the groups $G_p = G_{k_p}$ may be anisotropic. This case is precisely the essential part of the classical theorem of Eichler.

It is natural to ask whether we can obtain Eichler's theorem by our general method if we improve it somewhat.

Moreover, it was discovered that further investigation required certain other forms of groups of type A_n related to the anisotropic unitary groups.[1)] In this regard, the final part of the proof of the fundamental theorem in [1] requires revision.

The alterations needed in the proof of these cases turn out to be nonessential, and a similar argument works for arbitrary groups. In particular, we obtain a short new proof of Eichler's theorem.

2. Proof of the approximation theorem for groups G of type A_n. It is well known that for almost all p the group G_p is k_p-isotropic. We denote by T the (finite) set of those p for which G_p is anisotropic. As in [1], there is no loss in generality in restricting consideration to the case $S = \{\infty\}$. Let $V = \{p_1, p_2, \cdots, p_d\}$. Then by

1) I acknowledge Professor J.-P. Serre, who also discovered this.

Proposition 2 of [1], $\overline{\pi_V(G_{Z(V)})}$ is an open subgroup of finite index of the group $G_V = \Pi_{i=1}^d G_{p_i}$. By Theorem B of [1], the groups $G_p/Z(G_p)$, $p \notin T$, are abstract simple groups. Hence $G_p \subset \overline{G_\infty G_k}$ for $p \notin T$. Then $\pi_T(G_k) \subset \overline{G_\infty G_k}$. But $\overline{\pi_T(G_k)} = G_T$, since in this case the weak approximation theorem holds (see, for example, [3]). Thus $\overline{G_k G_\infty} = G_A$, which was to be shown.

3. In the formulation of the fundamental theorem in [1] we assumed that the finite set S contains $\{\infty\}$. This may have given the impression that this condition is essential.

It is, however, easy to show that this condition does not impose any loss of generality.

Indeed, it is sufficient to consider the case $S = \{p\}$, G_p noncompact. Let $T = \{\infty, p_1, \cdots, p_d\}$, where $p \neq p_i$, and let $V = \{p, p_1, \cdots, p_d\}$. We have to show that $\overline{\pi_T(G_{Z(V)})} = G_T = G_\infty \times \Pi_{i=1}^d G_{p_i}$. By a result in [1], $G_{Z(V)}$ is dense in $\Pi_{i=1}^d G_{p_i}$. It remains to prove that $G_\infty \subset \overline{\pi_T(G_{Z(V)})}$. This will follow from the fact that $G_{Z(p)}$ is dense in G_∞. Since G_p is noncompact, Borel's reduction theorem implies that G_Z has infinite index in $G_{Z(p)}$. Then it is not difficult to see that the connected component of the group $\overline{\pi_\infty(G_{Z(p)})}$ is nontrivial: if G is k-isotropic, $\overline{\pi_\infty(G_{Z(p)})}$ contains unipotent connected subgroups; otherwise G_∞/G_Z is compact and our assertion follows from the fact that G_Z is of infinite index in $G_{Z(p)}$. Since for $g \in G_k$ the groups $gG_{Z(p)}g^{-1}$ and $G_{Z(p)}$ are commensurable, the connected component of $\overline{\pi_\infty(G_{Z(p)})}$ is invariant with respect to G_k, hence also with respect to $G_\infty = \overline{\pi_\infty(G_k)}$. Since G_∞ is simple, $\overline{\pi_\infty(G_{Z(p)})} = G_\infty$. The latter equality implies that $\overline{\pi_T(G_{Z(V)})}$ is indiscrete in G_T; thus its connected component of the identity is nontrivial and invariant in G_T, i.e. it coincides with G_∞. This is what was required to be proved.

4. As an addition to the references given in [1] we mention the fundamental memoir of Weil [4] containing the solution to the approximation problem for many classical groups by analytic methods.

For a solution of the approximation problem over function fields which is analogous to the numerical case [1], an important contribution is the recent paper of Harder [5], which contains a proof of the functional analog of Borel's reduction theorem for adèle groups.

5. In conclusion, we correct one inaccuracy in the proof of Theorem B. Namely, on p. 1214 = 1142 it is stated that $\hat{H} = H \Rightarrow \Delta \cap H = (e)$. However, this is not always true, and for the unique non-quasi-decomposable groups of type E_6 and E_7 the proof requires the following additional argument. Suppose that $\Delta \cap H \neq (e)$. Examination of the diagram indices of the groups of type E_6 and E_7 reveals the existence of a simply connected k_v-decomposable subgroup M of type A_1 for E_6 and of type A_2 for E_7 such that $Z_G(H) \subset Z_G^0(M)M$, where $Z_G^0(M)$ is the connected component. It is not difficult to see that $Z_G^0(M) \cong A_5$. Consequently $Z_G^0(M) \cap M$ is a 2-group for E_6 and

a 3-group for E_7. It then follows from [1] that $G_{k_v}/G^u_{k_v}$ is a 2-group for E_6 and a 3-group for E_7. On the other hand, $\Delta \cap H$ must be a 3-group for E_6 and a 2-group for E_7. Consequently $G_{k_v}/G^u_{k_v}$ is a 3-group for E_6 and a 2-group for E_7. Taking these facts together, we see that $G_{k_v} = G^u_{k_v}$.

We also remark that on the same page: $H = \Pi^r_{i=1} h_{\alpha_i}(t_i)$ should read $H = \Pi^r_{i=1} h_{\bar{\alpha}_i}(t_i)$, where the $\bar{\alpha}_i$ are relative simple roots, and $H^{(k)} = \Pi^k_{i=1} h_{\alpha_i}(t_i)$ should read $H^{(k)} = \Pi^k_{i=1} \bar{h}_{\alpha_i}(t_i)$, where $\bar{h}_{\alpha_i}(t_i) = (\bigcap_{\alpha_j \neq \alpha_i} \operatorname{Ker} \bar{\alpha}_j)^0$, and $h_{\alpha_i}(t_i)$ should be replaced by $\bar{h}_{\alpha_i}(t_i)$ everywhere.

Received 10 FEB 1970

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Translated by:
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