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STATISTICAL MECHANICS OF THE COSMOLOGICAL MANY-BODY PROBLEM. II. RESULTS OF HIGHER ORDER CONTRIBUTIONS

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ABSTRACT

We calculate the irreducible triplet contribution to galaxy clustering in the cosmological many-body problem. These triplets generally represent short-lived configurations in which three close objects interact with pairwise gravitational forces. From the resulting grand canonical partition function, we obtain higher order analytical expressions for thermodynamic quantities such as specific heats, isothermal compressibility, and thermal expansion in the system. Compared with previous analyses, which included reducible but not irreducible triplets, the additional terms are usually small, especially in the limit of large *N*. This confirms the thermodynamic results. We also derive the modified spatial distribution functions and show that they agree with recent observational results. The inclusion of triplet and higher order irreducible clusters does not significantly modify the results of the reducible terms in the partition function or of its thermodynamic consequences such as the distribution function.

Subject headings: cosmology: theory — galaxies: clusters: general — gravitation — large-scale structure of universe — methods: analytical

1. INTRODUCTION

The discovery of a fundamental statistical mechanical description of the cosmological many-body problem (Ahmad et al. 2002, hereafter Paper I) has provided a more rigorous basis for its earlier thermodynamic description (summarized, e.g., in Saslaw 2000). The statistical mechanical approach agrees with thermodynamics and is also more powerful for some purposes such as determining the velocity and energy distribution functions, along with the associated probabilities that clusters are bound and virialized (Leong & Saslaw 2004). It is also more general, allowing the particles to have softened gravitational potentials, corresponding to models of galaxies with dark matter halos. Furthermore, it can be generalized to systems in which the particles, such as galaxies, have different masses (Ahmad et al. 2006). These results agree very well with *N*-body simulations (Itoh et al. 1988, 1990, 1993; Inagaki et al. 1992). They also describe the observed spatial distribution functions of galaxies with increasing accuracy as the catalogs become larger and more complete. The most recent example is the analysis of the extended sources in the 2MASS (Two Micron All Sky Survey) catalog, where the theory, which has no free parameters, agrees with the observations to better than 97% (Sivakoff & Saslaw 2005).

The cosmological many-body problem simply asks how a very large number of particles will cluster under their mutual gravitational interactions in a statistically homogeneous, slowly expanding universe. These last two conditions imply that over a wide range of spatial scales the system is in quasi-equilibrium. Then, the usual thermodynamic quantities such as density, pressure, temperature, and correlation energy are well defined and relax locally at a faster rate than they change globally due to the universe's expansion. This is reasonable theoretically in the usual models of general relativity (Saslaw & Fang 1996; Saslaw 2000), as well as in those currently modified by dark energy or quintessence. Details of these models affect the amplitude of clustering as a function of time or redshift, but they do not change the form of the distribution functions, provided they satisfy the quasi-equilibrium condition. The theory applies if the dark matter is mostly in halos around individual galaxies when they cluster, or if dark matter and energy are distributed essentially uniformly.

A detailed description of the conditions and length scales for quasi-equilibrium evolution remains an important unsolved problem; *N*-body simulations (e.g., Itoh 1990) showed that a range of initial states whose power spectra are power laws with exponents between about ± 1 satisfy quasi-equilibrium evolution. They agree with the present observations of the galaxy spatial distribution. A characteristic feature of these simulations is that they have no initial nonlinear structures that extend over scales comparable with the system and that are stable for at least several expansion timescales. Such initial states would not relax to a quasi-equilibrium state, and the distribution functions of systems containing them would not evolve into the form of equation (58) below. This type of unrelaxed initial structure occurs in some CDM (cold dark matter) models (Fukushige & Makino 2003), and we would not expect such examples to be described well by equation (58). Other examples, with low-amplitude initial structure on large scales, can relax (cf. Saslaw 2000, p. 372) to equation (58). Unfortunately, most CDM models require a number of additional assumptions to convert their dark matter distribution to visible galaxies. The consequences of these assumptions for more powerful statistics such as spatial and velocity distribution functions have not yet been systematically examined. The cosmological many-body system explored here is relatively more straightforward, and its physical description is also of interest in its own right.

Our analysis in Paper I started by deriving the grand canonical partition function, from which all the basic statistical mechanical and thermodynamic properties of this system can be calculated. To simplify the analytic derivation in Paper I, we neglected terms in the configuration integral (eq. [9] of Paper I or eq. [3] below) of the form $f_{12}f_{13}f_{23}$ ($\triangle$), and higher order irreducible terms. These represent the simultaneous pairwise (because the individual gravitational potentials always depend on just two particles) interactions of three or more particles. Even though early results (Saslaw et al. 1990) suggested these irreducible terms would be small, their significance and consequences need to be investigated for several reasons.

First, it is a check on the validity of the results in Paper I and how they would be modified by this more exact calculation. Second, it has long been known that the galaxy three-point correlation function can be reasonably well approximated by the form

$$\xi_3 = Q_3(\xi_{12}\xi_{13} + \xi_{23}\xi_{21} + \xi_{31}\xi_{32}),$$

which does not include the triplet term $\xi_{123}(r_{12}, r_{23}, r_{31})$. Observed values of Q_3 are given as 0.80 ± 0.07 (Groth & Peebles 1977) and 1.0 ± 0.2 (Peebles 1993) for optical samples, with possibly smaller values for infrared samples. In the three-dimensional Southern Sky Redshift sample, Benoist et al. (1999) found $Q_3 = 0.61 \pm 0.04$. Their analysis gives a range of Q_3 values depending on the subsample. These are roughly consistent with the value of 0.75 (Saslaw et al. 1990) derived for the thermodynamic description of the cosmological many-body problem. Since ξ_{123} is determined by the same class of spatially close three-body interactions described by $f_{12}f_{13}f_{23}$, our calculation of the latter will also imply that ξ_{123} is relatively unimportant, which has previously been a bit mysterious. Third, our understanding of the irreducible triplet term provides insight into the role of strong three-body interactions, which, depending on initial conditions, may provide nuclei that influence further clustering. Fourth, our gravitational system shows interesting differences from the more familiar case of a molecular or atomic gas. In the gaseous case, the irreducible triplet terms lead to additional polarization effects (Graben & Present 1962), which have been computed by several methods (Axilrod 1951; Midzuno & Kihara 1956). An elegant geometric method by Katsura (1959) is useful for low-order irreducible contributions for hard-core and square-well potentials, and fourth and higher order terms can be estimated numerically with Monte Carlo techniques (e.g., Alder & Hoover 1968). However, for the gravitational potential, we must make a fresh start from first principles.

In § 2 we introduce the basic cluster expansion formalism and our notation. Then we evaluate the irreducible triplet term and use it to derive the partition function to a higher order approximation in § 3. In § 4 we examine the effects of irreducible triplets on the thermodynamic functions, and then in § 5 we use these results to derive the modified spatial distribution function and compare it with recent observations. Section 6 summarizes and discusses the results further.

2. BASIC FORMALISM

As a first step toward the application of the cluster expansion method to the gravitational many-body problem, we consider a single-component, classical system whose potential energy is given by a sum of two-particle interactions. The general partition function for a system of N particles of mass m, average temperature T, and momenta p_i interacting via a pair potential ϕ is

$$\begin{aligned} Z_N(T, V) = &\frac{1}{\Lambda^{3N} N!} \\ &\times \int \exp\left\{-\left[\sum_{i=1}^{N}\frac{p_i^2}{2m} + \phi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)\right]T^{-1}\right\} d^{3N}\mathbf{p}\, d^{3N}\mathbf{r} \\ = &\frac{1}{N!}\left(\frac{2\pi m T}{\Lambda^2}\right)^{3N/2} Q_N(T, V), \end{aligned} \quad (1)$$

where Λ normalizes the phase space volume cell and

$$Q_N(T, V) = \int \ldots \int \exp\left[-\phi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)T^{-1}\right] d^{3N}\mathbf{r} \quad (2)$$

is the usual configuration integral, whose evaluation is very complicated, involving a $3N$-fold volume integration. However, for the cosmological many-body problem, Paper I shows that the partition function has a relatively simple solution for reasonable physical approximations. Introducing the Mayer function f_{ij}, the configuration integral is given by

$$Q_N(T, V) = \int \ldots \int \prod_{1 \le i < j \le N} \left(1 + f_{ij}\right) d^{3N}\mathbf{r}, \quad (3)$$

where

$$f_{ij} = e^{(-\phi_{ij}/T)} - 1. \quad (4)$$

The integrand in equation (3) can be expanded by writing

$$\begin{aligned} W_N(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) &= \prod_{1 \le i < j \le N} (1 + f_{ij}) \\ &= 1 + \sum_{1 \le i < j \le N} f_{ij} + \sum f_{ij} f_{i'j'} + \sum f_{ij} f_{i'j'} f_{i''j''} + \ldots. \end{aligned} \quad (5)$$

Each set of summations corresponds to a subdivision of the particles into a particular type of group. Each summed term of the series in equation (5) is a sum over all the coupled products of the same order N. This total summation can be represented by all topologically different N-particle diagrams. The first few values of N give

$$W_1(\mathbf{r}_1) = 1, \quad (6)$$

$$W_2(\mathbf{r}_1, \mathbf{r}_2) = (1 + f_{12}), \quad (7)$$

$$W_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = (1 + f_{12})(1 + f_{13})(1 + f_{23}), \quad (8)$$

$$\begin{aligned} W_4(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) = &(1 + f_{12})(1 + f_{13})(1 + f_{14}) \\ &\times (1 + f_{23})(1 + f_{24})(1 + f_{34}). \end{aligned} \quad (9)$$

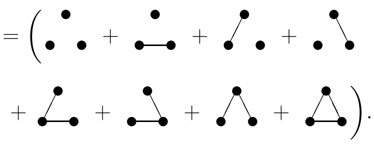
If we expand the expression for $W_N(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$, we find that the case $N = 1$ contains only one term, the case $N = 2$ contains two terms, the case $N = 3$ contains eight terms, the case $N = 4$ contains 64 terms, and so on.

Since the grand canonical partition function contains terms with arbitrarily large values of N, it is necessary to find a systematic procedure for categorizing the various terms in the expansion of W_N. With the usual graph theory procedure, we can represent $W_N(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$ as

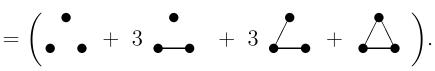
$$W_N(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) = \sum (\text{all different } N\text{-particle graphs}). \tag{10}$$

An N-particle graph is obtained by drawing N numbered circles and connecting them together, with the number of lines (including zero) representing the interactions in equation (5), subject only to the condition that no two circles are connected by more than one line. Two N-particle graphs are different if the numbered circles are connected differently.

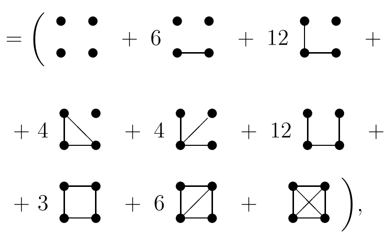
The case $N = 3$ has eight terms, which can individually be written, diagrammatically, as

$W_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) =$  (11)

This can be represented by grouping all graphs with the same topological structure together, reducing the graph expression to

$W_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) =$  (12)

Similarly, the 64 terms corresponding to $N = 4$ can be grouped together so that

$W_4(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) =$  (13)

and so on.

There is another reason for writing the configurational integral as an expansion of graphs, which will become apparent when we actually evaluate $Q_N(T, V)$. For this, the following properties will be useful:

1. Each unconnected graph gives a factor of V.
2. If two parts of a graph are connected by a line, the corresponding algebraic expression factors into a product.
3. Each completely connected part of a graph is proportional to the volume.
4. Some special graphs with an index in common decompose into a product.

Using the above properties, it can be shown that the terms in the expansion can be rearranged in such a manner that it is not necessary to calculate all clusters afresh, but only certain types known as irreducible clusters. These irreducible clusters have at least two independent paths, not crossing at a node, between each pair of particles. Thus, the rightmost graph in equation (12) and three last graphs in equation (13) are irreducible, as is the graph for W_2, by definition.

We now return to the evaluation of the configuration integral, using the method of Paper I. In this approach, all interactions were taken with respect to an arbitrary particle (say, particle 1) assumed to be at the origin of the coordinate system so that with the exclusion of self-energy terms of the form f_{jj}, equation (3) became

$$Q_N(T, V) = \int \ldots \int \prod_{1 \le i < j \le N} (1 + f_{12})(1 + f_{13}) \ldots (1 + f_{1N})\, d^3\mathbf{r}_1\, d^3\mathbf{r}_2 \ldots d^3\mathbf{r}_N \tag{14}$$

in Paper I. Note that for $N = 2$, we had only one term $(1 + f_{12})$; for $N = 3$, we had $(1 + f_{12})(1 + f_{13})$; and so on for large N. Furthermore, one can see that the product term $(1 + f_{23})$ was missing because all interactions are taken with respect to particle 1, and f_{23} does not involve particle 1. Therefore, the triplet interactions of the form $f_{12} f_{13} f_{23}$ were neglected in Paper I. Now, however, we examine their effect in § 3. The configurational integral ignoring the irreducible triplet term for point-mass particles, derived in Paper I, was

$$Q_N(T, V) = V^N \left(1 + \beta \bar{n} T^{-3}\right)^{N-1}, \tag{15}$$

where

$$\beta = 3(Gm^2)^3/2 \tag{16}$$

and $\bar{n} = \bar{N}/V$ is the average number density of a volume V in the ensemble. Equation (15) has a simple form for non–point-mass particles, also derived in Paper I.

3. THE PARTITION FUNCTION INCLUDING THE IRREDUCIBLE TRIPLET INTERACTION

In the previous work we calculated the partition function including the doublet irreducible graph (•—•) and the reducible triplet graphs in equation (12). The earlier analysis neglected terms of the form $f_{12} f_{13} f_{23}$ (△), and higher order irreducible terms, which represent the simultaneous pairwise interaction of three or more particles. The physical reason for neglecting the (△) terms was that close configurations of three particles occur infrequently and, whether bound or not, are unstable and last for relatively short times. Besides, they are more complicated to calculate. Now that we are able to calculate these terms, we can explore their influence and the accuracy of our previous assumptions.

The inclusion of the irreducible triplet term (△) modifies the partition function and hence all the other thermodynamic quantities, as well as the distribution function. Thus, to calculate this more exact partition function, we first evaluate the triplet term,

$$I = \int \int \int f_{12} f_{13} f_{23}\, d^3\mathbf{r}_1\, d^3\mathbf{r}_2\, d^3\mathbf{r}_3. \tag{17}$$

3.1. *Evaluation of the Triplet Term*

Let particle 1 be at the origin of the coordinate system (Fig. 1), which is also the center of a spherical cell of radius R_1. If the

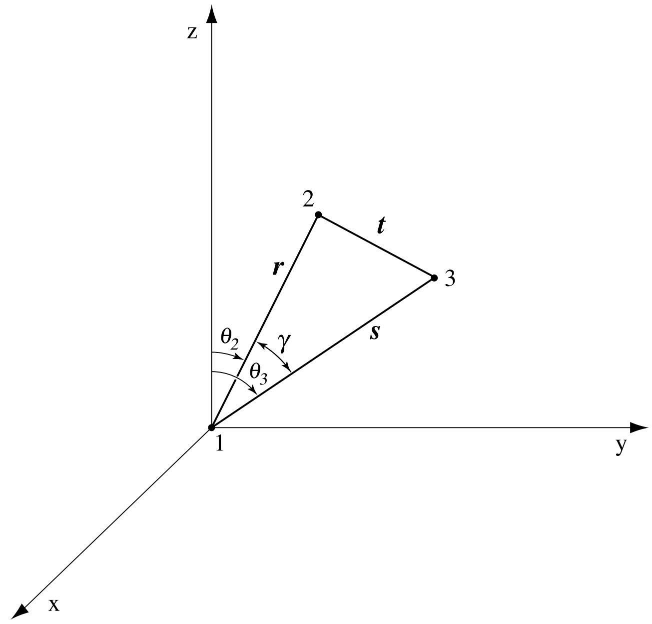


FIG. 1.—Coordinate system for the irreducible triplet term.

coordinates of particle 2 are $(\mathbf{r}, \theta_2, \phi_2)$ and of particle 3 are $(\mathbf{s}, \theta_3, \phi_3)$ and the vector displacement between them is $\mathbf{t}$, then we can write for any temperature T

$$f_{12} = \frac{Gm^2}{T|\mathbf{r}_{12}|} = \frac{Gm^2}{Tr}, \quad (18)$$

$$f_{13} = \frac{Gm^2}{T|\mathbf{r}_{13}|} = \frac{Gm^2}{Ts}, \quad (19)$$

$$f_{23} = \frac{Gm^2}{T|\mathbf{r}_{23}|} = \frac{Gm^2}{Tt}. \quad (20)$$

Note that for simplicity we do not introduce the softening parameter, ϵ, into the gravitational potential here, because Paper I showed that the relevant integrals converge uniformly as $\epsilon \to 0$, and in any case ϵ produces a small physical effect if $\epsilon/R_1 \lesssim 0.5$.

Therefore,

$$\begin{aligned}\int\int\int f_{12} f_{13} f_{23}\, d^3\mathbf{r}_1\, d^3\mathbf{r}_2\, d^3\mathbf{r}_3 &= V\int\int f_{12} f_{13} f_{23}\, d^3\mathbf{r}_{12}\, d^3\mathbf{r}_{13} \\ &= V\int\int \left(\frac{Gm^2}{T}\right)^3 \frac{1}{rst}\, d^3\mathbf{s}\, d^3\mathbf{r}. \end{aligned} \quad (21)$$

The corresponding volume element is

$$d^3\mathbf{r}_{12}\, d^3\mathbf{r}_{13} = d^3\mathbf{r}\, d^3\mathbf{s} = s^2\, ds \sin\theta_3\, d\theta_3\, d\phi_3 r^2\, dr \sin\theta_2\, d\theta_2\, d\phi_2. \quad (22)$$

In addition,

$$t^2 = r^2 + s^2 - 2rs\cos\gamma, \quad (23)$$

where $\gamma = \theta_3 - \theta_2$ is the angle between $\mathbf{r}$ and $\mathbf{s}$ given by

$$\cos\gamma = \cos\theta_3\cos\theta_2 + \sin\theta_3\sin\theta_2\cos(\phi_3 - \phi_2). \quad (24)$$

Substitution of equation (23) into equation (21) gives

$$\int\int\int f_{12} f_{13} f_{23}\, d^3\mathbf{r}_1\, d^3\mathbf{r}_2\, d^3\mathbf{r}_3 = V\left(\frac{Gm^2}{T}\right)^3 \times \int_V\int_V \frac{d^3\mathbf{r}\, d^3\mathbf{s}}{rs(r^2 + s^2 - 2rs\cos\gamma)^{1/2}}. \quad (25)$$

The denominator of equation (25) can be written in terms of Legendre polynomials,

$$\frac{1}{(r^2 + s^2 - 2rs\cos\gamma)^{1/2}} = \begin{cases} \sum_{n=0}^{\infty} P_n(\cos\gamma)\frac{s^n}{r^{n+1}}, & \text{for } s < r, \\ \sum_{n=0}^{\infty} P_n(\cos\gamma)\frac{r^n}{s^{n+1}}, & \text{for } s > r. \end{cases} \quad (26)$$

Using the addition theorem, we can write this in terms of associated Legendre polynomials,

$$\begin{aligned}P_n(\cos\gamma) &= P_n(\cos\theta_3)P_n(\cos\theta_2) \\ &+ 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta_3)P_n^m(\cos\theta_2)\cos(\phi_3 - \phi_2).\end{aligned} \quad (27)$$

To simplify the calculations, we consider particle 2 to be at the polar axis, such that $\theta_2 = 0$, so that $\gamma = \theta_3$. We have two cases, $s < r$ and $s > r$, both lying within the sphere of radius R_1.

If we consider the case $s < r$, then the limits of integration for s are from 0 to r and for r are from 0 to R_1, so that equation (25) becomes

$$\begin{aligned}\int\int\int f_{12} f_{13} f_{23}\, d^3\mathbf{r}_1\, d^3\mathbf{r}_2\, d^3\mathbf{r}_3 = \\ V\left(\frac{Gm^2}{T}\right)^3 \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \frac{s\, ds \sin\theta_3\, d\theta_3\, d\phi_3}{(r^2 + s^2 - 2rs\cos\theta_1)^{1/2}} \\ \times \int_0^{2\pi}\int_0^{\pi}\int_0^{R_1} \frac{r^2\, dr \sin\theta_2\, d\theta_2\, d\phi_2}{r}.\end{aligned} \quad (28)$$

First, we integrate over all the angles using equations (26) and (27), and then integrate over s and finally integrate over r to obtain

$$\begin{aligned}\int\int\int f_{12} f_{13} f_{23}\, d^3\mathbf{r}_1\, d^3\mathbf{r}_2\, d^3\mathbf{r}_3 &= \frac{8\pi^2 V}{3}\left(\frac{Gm^2}{T}\right)^3 R_1^3 \\ &= \frac{3}{2}V^3\left(\frac{Gm^2}{TR_1}\right)^3.\end{aligned} \quad (29)$$

It can similarly be shown that the case $s > r$ leads to the same result as equation (29). Incorporating equation (16) gives

$$\int\int\int f_{12} f_{13} f_{23}\, d^3\mathbf{r}_1\, d^3\mathbf{r}_2\, d^3\mathbf{r}_3 = \frac{4}{9}V^3\left(\beta\bar{n}T^{-3}\right)^3. \quad (30)$$

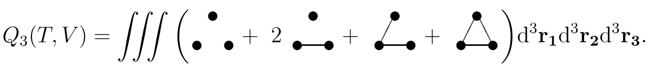
This simple result for a complicated integral allows us to readily determine the contribution of (△) terms to the partition function.

3.2. *Evaluation of the Configuration Integral*

For a triplet, the minimum requirement is three interacting particles, so it will not influence the expressions for $Q_1(T, V)$ and $Q_2(T, V)$. However, for $N = 3$, we have a single irreducible triplet contribution, so that

$$Q_3(T,V) = \int\int\int (1+f_{12})(1+f_{13})\, d^3\mathbf{r}_1\, d^3\mathbf{r}_2\, d^3\mathbf{r}_3 + \int\int\int f_{12}f_{13}f_{23}\, d^3\mathbf{r}_1\, d^3\mathbf{r}_2\, d^3\mathbf{r}_3. \quad (31)$$

In terms of cluster diagrams, equation (31) can be written as



(32)

It may be noted from equation (32) that we have followed our earlier analysis by taking all interactions with respect to particle 1, including the triplet term of the form (△) in which particles 2 and 3 are also connected through particle 1. Since particle 1 could be any of the N particles, this produces no loss of generality.

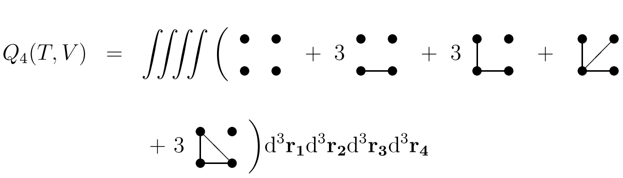
Using equations (15) and (30), we therefore have

$$Q_3(T,V) = V^3\left[\left(1+\beta\bar{n}T^{-3}\right)^2 + 1\times\frac{4}{9}\left(\beta\bar{n}T^{-3}\right)^3\right]. \quad (33)$$

Now, for $N = 4$, there will be a total of four irreducible isolated triplets (i.e., unconnected to other particles). In fact, for any N, the number of irreducible isolated triplets is given by $N(N-1)(N-2)/6$. However, since our analysis requires that particle 1 should necessarily be involved in all the interactions, we have only three such triplets for $N = 4$. Consequently,

$$Q_4(T,V) = \int\int\int\int (1+f_{12})(1+f_{13}) \times (1+f_{14})\, d^3\mathbf{r}_1\, d^3\mathbf{r}_2\, d^3\mathbf{r}_3\, d^3\mathbf{r}_4 + 3\int\int\int\int f_{12}f_{13}f_{23}\, d^3\mathbf{r}_1\, d^3\mathbf{r}_2\, d^3\mathbf{r}_3\, d^3\mathbf{r}_4. \quad (34)$$

In terms of cluster diagrams, this becomes



$$= V^4\left\{(1+\beta\bar{n}T^{-3})^3 + 3\times\frac{4}{9}(\beta\bar{n}T^{-3})^3\right\}. \quad (35)$$

Note that each unconnected dot, on integration, gives a factor of V. For $N = 5, 6, 7, \ldots$, we have 6, 10, 15, ... isolated irreducible triplets, so that, in general, we have only $(N-1)(N-2)/2$ $(N \geq 2)$ such triplets.

The general configuration integral for $N \geq 3$ incorporating the triplet terms therefore takes the form

$$Q_N(T,V) = V^N\left[\left(1+\beta\bar{n}T^{-3}\right)^{N-1} + \frac{(N-1)(N-2)}{2}\frac{4}{9}\left(\beta\bar{n}T^{-3}\right)^3\right]. \quad (36)$$

As a result, the partition function, equation (1), is calculated for $N \geq 3$ to be

$$Z_N = \frac{V^N}{N!}\left(\frac{2\pi mT}{\Lambda^2}\right)^{3N/2} \times\left[\left(1+\beta\bar{n}T^{-3}\right)^{N-1} + \frac{(N-1)(N-2)}{2}\frac{4}{9}\left(\beta\bar{n}T^{-3}\right)^3\right]. \quad (37)$$

The second term on the right-hand side of equation (37) arises as a result of the inclusion of irreducible triplet interactions. It is trivial to see that the above equation reduces to the same form of the partition function as was obtained previously in Paper I if the triplet contributions are neglected. We see that for $\beta\bar{n}T^{-3} < 1$, the first term in equation (37) dominates, as it does for large N even if $\beta\bar{n}T^{-3} > 1$.

4. THERMODYNAMIC FUNCTIONS

It is clear that the partition function is modified by the higher order terms. Since all the thermodynamic quantities are derivable from the partition function, these will be modified accordingly for $N \geq 3$.

We start with the free energy, since it is the fundamental quantity for evaluating other thermodynamic quantities,

$$F = -T\ln Z_N(T,V) = NT\ln\left(\frac{NT^{-3/2}}{V}\right) - NT\ln\left(1+\beta\bar{n}T^{-3}\right) - \frac{3}{2}NT\ln\left(\frac{2\pi m}{\Lambda^2}\right) - T\ln\left[1+\frac{a_N(\beta\bar{n}T^{-3})^3}{(1+\beta\bar{n}T^{-3})^N}\right] - NT, \quad (38)$$

where

$$a_N \equiv 2(N-1)(N-2)/9. \quad (39)$$

The entropy of the system can be derived from the free energy as

$$S = -\left(\frac{\partial F}{\partial T}\right)_{N,V} = -N\ln\left(\frac{NT^{-3/2}}{V}\right) + N\ln\left(1+\beta\bar{n}T^{-3}\right) + \ln\left[1+\frac{a_N(\beta\bar{n}T^{-3})^3}{(1+\beta\bar{n}T^{-3})^N}\right] - 3N\frac{\beta\bar{n}T^{-3}}{1+\beta\bar{n}T^{-3}} + \frac{5}{2}N + \frac{3}{2}N\ln\left(\frac{2\pi m}{\Lambda^2}\right) + \frac{3a_N(\beta\bar{n}T^{-3})^3}{(1+\beta\bar{n}T^{-3})}\left[\frac{N(\beta\bar{n}T^{-3}) - 3(1+\beta\bar{n}T^{-3})}{a_N(\beta\bar{n}T^{-3})^3 + (1+\beta\bar{n}T^{-3})^N}\right]. \quad (40)$$

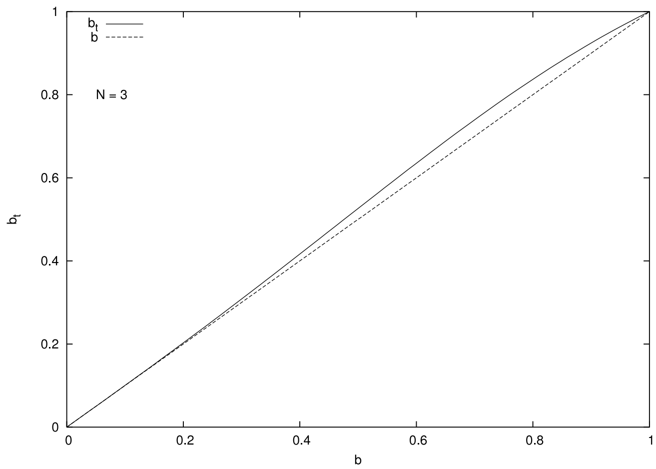


FIG. 2.—Comparison of b_t and b for fixed values of N.

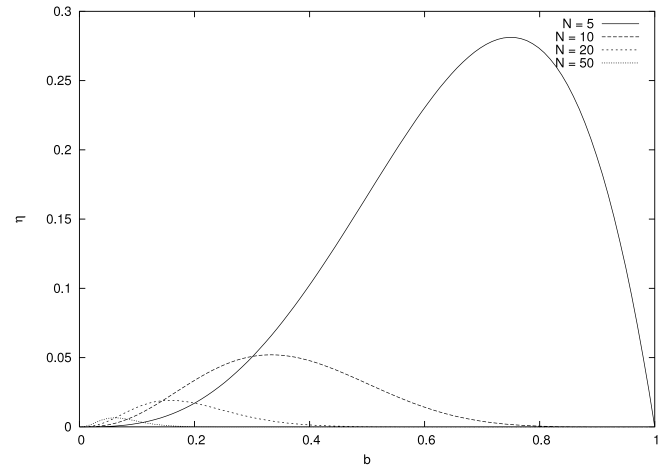


FIG. 3.—Variation of η as a function of b for various values of N. For larger values of N, η tends to zero.

The internal energy has the form

$$U = F + TS = \frac{3}{2} NT(1 - 2b_t). \tag{41}$$

Here, the irreducible triplet clustering parameter b_t is defined for $N \geq 3$ by

$$b_t = b\left[\frac{N + 3a_N b^2(1-b)^{N-3}}{N + a_N N b^3 (1-b)^{N-3}}\right], \tag{42}$$

$$b = \frac{\beta \bar{n} T^{-3}}{1 + \beta \bar{n} T^{-3}}. \tag{43}$$

The form of b_t as given by expression (42) incorporates the effects of triplet interactions. The values of b_t satisfy the same general boundary conditions as b. For example, $b_t \rightarrow 0$ as $b \rightarrow 0$, and $b_t \rightarrow 1$ as $b \rightarrow 1$, which implies that b_t is simply a modified form of b in which the modification arises from the triplet terms of the form (△). Figure 2 shows that the inclusion of irreducible triplets has a small effect on b just for $N = 3$. For larger values of $N \geq 5$, the distinction between b and b_t is smaller. Since both b and b_t are measures of gravitational attraction, we can infer that the clustering may not be significantly affected by the inclusion of the triplet interactions, especially as $b \rightarrow 1$ and for large N. Pressure can be calculated from

$$P = -\left(\frac{\partial F}{\partial V}\right)_{N,T} = \frac{NT}{V}(1 - b_t). \tag{44}$$

The chemical potential can be evaluated by using the free energy and pressure equation of state from

$$\frac{N\mu}{T} = \frac{F}{T} + \frac{PV}{T} \tag{45}$$

to give

$$e^{N\mu/T} = \left(\frac{\bar{N}}{V} T^{-3/2}\right)\left(\frac{2\pi m}{\Lambda^2}\right)^{-3N/2} \frac{(1-b)^N}{\left[1 + a_N b^3 (1-b)^{N-3}\right]} e^{-Nb_t}. \tag{46}$$

Another important property of a system is the behavior of the specific heat as a function of the clustering parameter b, which determines the stability of the system. By definition,

$$C_V = \frac{1}{N}\left(\frac{\partial U}{\partial T}\right)_{N,V}. \tag{47}$$

Using equation (42),

$$C_V = \frac{3}{2}\left(1 - 2b_t - 2T\frac{\partial b_t}{\partial T}\right). \tag{48}$$

After some algebra, we obtain

$$C_V = \frac{3}{2}[1 - 2b\delta_1 + 6b(1-b)\delta_2], \tag{49}$$

where δ_1 and δ_2 are defined as

$$\delta_1 = \frac{1 + 3N^{-1}(1-b)b^{-1}\eta}{1 + (1-b)\eta}, \tag{50}$$

$$\delta_2 = \frac{1 - [2(1-b)b^{-1}N^2 + 3N(4+b) - 9]N^{-2}\eta}{[1 + (1-b)\eta]^2}, \tag{51}$$

with η given by

$$\eta(N, b) = \begin{cases} a_N b^3 (1-b)^{N-4}, & \text{for } N \geq 3, \\ 0, & \text{for } N < 3. \end{cases} \tag{52}$$

Figure 3 illustrates $\eta(N, b)$ as a function of b for given values of the total number of particles N. Figure 4 shows the behavior of the specific heat given by equation (49) (*solid line*) as a function of b. Note that here we are considering an ensemble of systems in which each system has N particles. The result of Saslaw & Sheth (1993) without irreducible triplet contributions is also plotted (*dotted line*), for comparison. For $N = 1$ and $N = 2$, the two curves coincide, since there are no triplets. For $N = 3$, there is only one triplet, and the two curves overlap for lower values of b, but for $0.4 \leq b \leq 1$, the triplets lower the specific heat. As

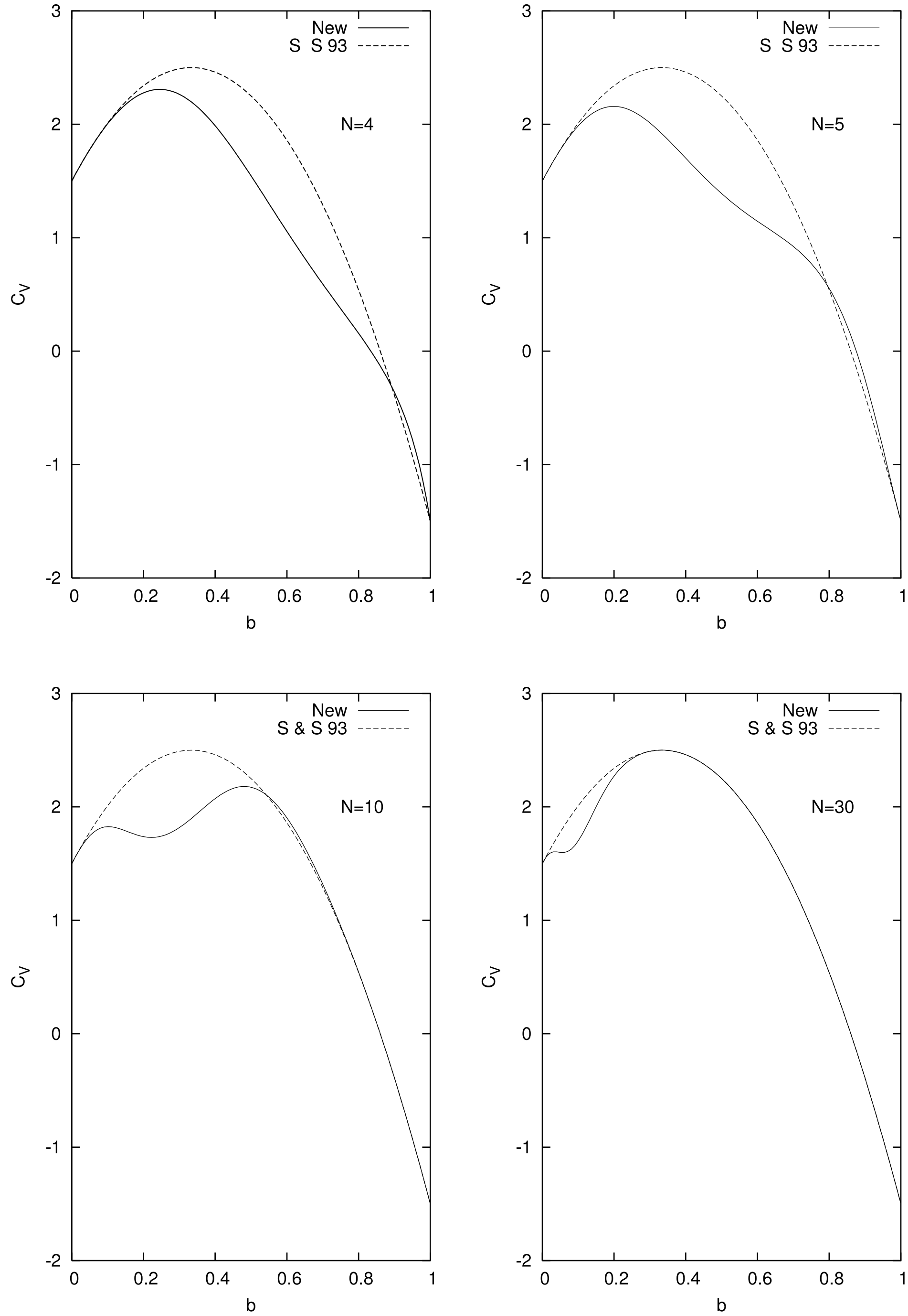


FIG. 4.—Comparisons of specific heat. The solid line is our result from eq. (49), and the dashed line is taken from Saslaw & Sheth (1993). Note that for $N = 1$ and 2 the two curves coincide, as expected, and for larger values of N the agreement becomes closer.

N increases, the decrease in the specific heat becomes appreciable for intermediate values of b for which triplets are the important clusters in the system. For $N \gtrsim 50$, the two curves nearly coincide, since larger clusters dominate. This decrease in specific heat for lower values of N and intermediate values of b is due to the contribution of irreducible triplets. As discussed earlier (Saslaw & Sheth 1993; Saslaw 2000), the effect of triplets in general is to increase gravitational binding and to decrease the specific heat. For lower values of N, there is an appreciable number of triplets compared to the doublets, so that the effect is enhanced. For large values of N and lower values of $b \lesssim 1/3$, the doublets still remain as the dominant components, so that the effect is less significant. However, for $b = 0$, i.e., when the galaxies are uncorrelated, the system behaves as a perfect monatomic gas ($C_V = 3/2$), as before. Similarly, the value of $C_V = -3/2$ as $b \to 1$ results from the system becoming virialized on all scales and is not much affected by irreducible triplets. Thus, we see that the behavior of specific heat as given by equation (49) is understandable and agrees with the earlier results (Saslaw & Sheth 1993).

Two other thermodynamic properties, important for the stability of the systems, are just stated here. These are the compress-

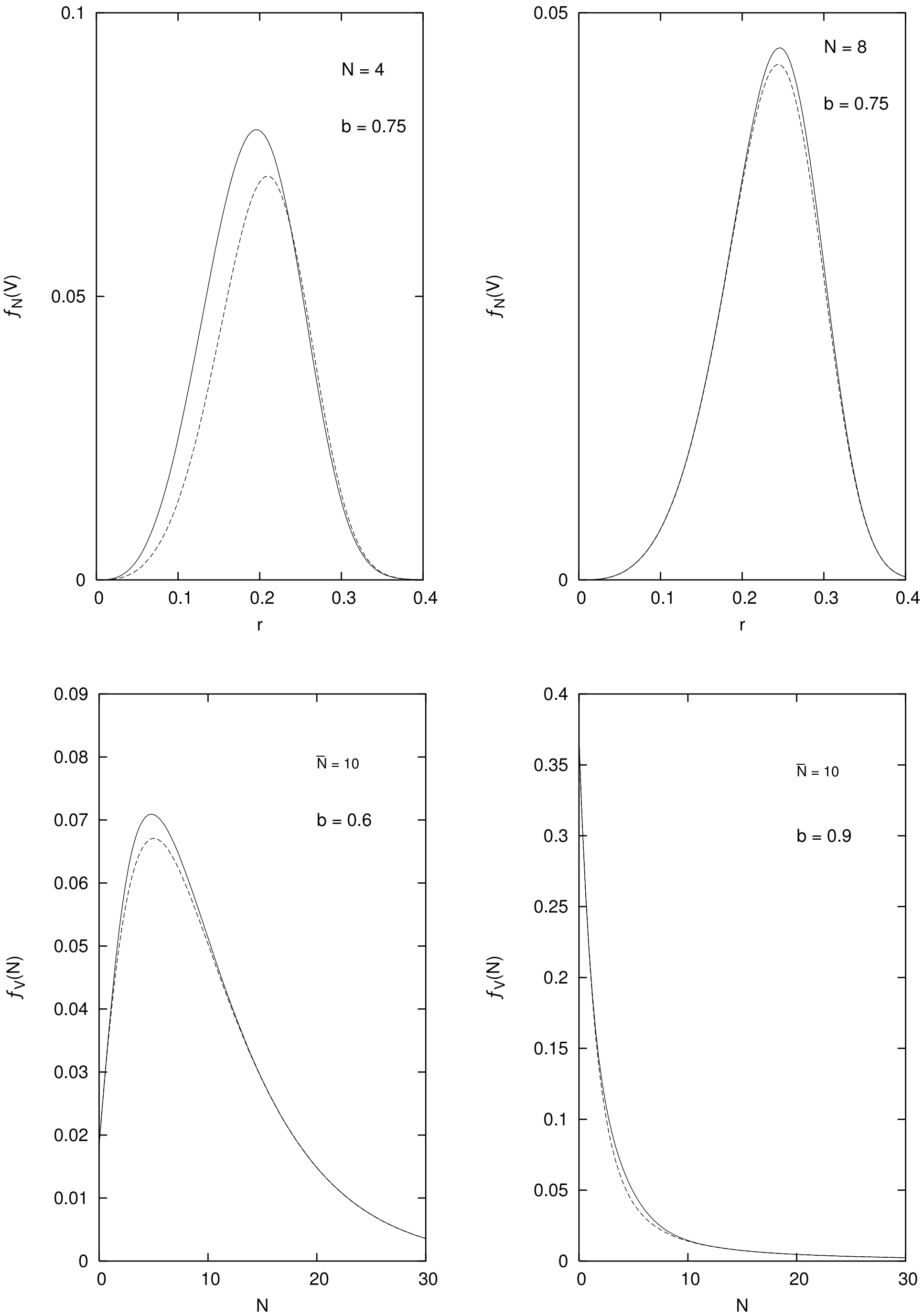


FIG. 5.—Distribution function $f_N(V)$ and $f_V(N)$ for a system that includes the triplet interactions (*solid lines*) compared with the case excluding irreducible triplets (*dashed lines*).

ibility at constant temperature, κ_T, and the coefficient of thermal expansion, γ:

$$\gamma = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_{P,N} = \frac{1}{T}\left(\frac{1+3b}{1-b}\right)\left\{\frac{1+(1-3/N)(1+3b)^{-1}[b(4N-1)-11]\eta}{1+(1-3/N)\eta} + \frac{4(1-b)(3/N-b)\eta}{(1+3b)[1+(1-b)\eta]}\right\}, \quad (53)$$

$$\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{T,N} = \frac{V}{NT(1-b)^2} \times \left\{\frac{1+(1-b)\eta+[b(N-1)-2][1+(1-b)\eta](1-3/N)\eta}{[1+(1-3/N)(1-b)\eta]^2} + \frac{(3/N-b)(1-b)\eta}{1+(1-3/N)\eta}\right\}. \quad (54)$$

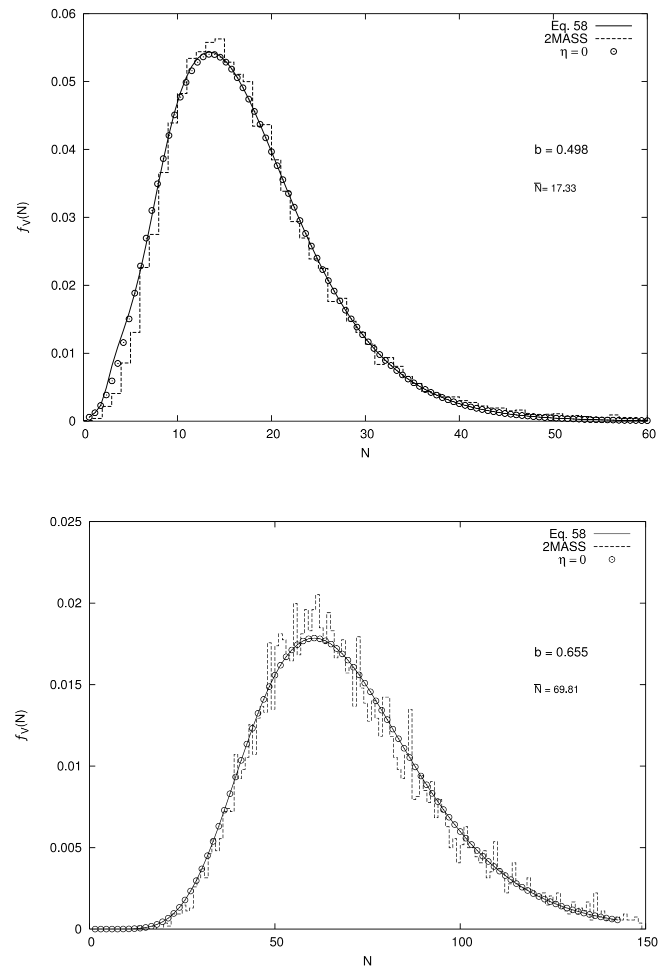


Fig. 6.—Comparison of the observed galaxy distribution function taken from the 2MASS survey (*histograms*; Figs. 2*d* and 2*e* of Sivakoff & Saslaw 2005) with and without the irreducible triplet contributions.

The specific heat at constant pressure is related to above quantities by

$$C_P = C_V + \frac{TV\gamma^2}{N\kappa_T}. \quad (55)$$

Writing these results in this rather elegant form makes it clear that for small values of η, implied either by large N or by small b or by b nearly unity, the quantities in curly brackets are nearly unity and γ, κ_T, and C_P all reduce to their values without irreducible triplets.

5. DISTRIBUTION FUNCTIONS

The galaxy distribution function provides useful insights into the observed galaxy clustering, as well as into results of N-body simulations. Hence, we examine how irreducible triplets contribute to it. The distribution function $f(N)$ is derived from its general relation to the grand canonical partition function Z_G, the chemical potential μ, and the sum over energy states Z_N,

$$f(N) = \frac{e^{N\mu/T} Z_N(T, V)}{Z_G(T, V, \mu)}, \quad (56)$$

where Z_G is related to the equation of state by

$$\ln Z_G = \frac{PV}{T} = \bar{N}(1 - b_t), \quad (57)$$

similar to the result in Paper I. From equations (42)–(46), for any particular value of N, the distribution function becomes (note that here N is not the total number of particles in the system)

$$f(N) = \frac{\bar{N}(1-b)\left[\bar{N}(1-b) + Nb\right]^{N-1} + N^3\left(\bar{N}\right)^{N-3}(1-b)\eta}{N![1 + (1-b)\eta]} \times e^{-Nb_t - \bar{N}(1-b_t)}. \quad (58)$$

Figure 5 shows the distribution function $f_N(V)$ for fixed values of N and b. One can see that for smaller values of N, the two graphs differ. The peak corresponding to the distribution function of equation (58) is higher and slightly shifted toward lower r-values. As N increases, this distinction begins to disappear, and even for not so high values of N, the two graphs overlap. Figure 5 also shows the distribution function $f_V(N)$ for $\bar{N} = 10$ and different values of b. For $b = 0.1$, indicating negligible correlations, $f_V(N)$ is not affected by the inclusion of triplets, since this system behaves almost as a perfect gas. As b increases ($b = 0.6$), the inclusion of irreducible triplets increases the peak of the distribution and shifts it slightly toward lower N. As b increases further ($b = 0.9$), the effects of irreducible triplets diminish. Equations (42) and (52) also show how irreducible triplets modify the void distribution $f_0(V)$.

We next examine the effects of irreducible triplets on the comparison with the recently observed galaxy distribution functions in Figures 2*d* and 2*e* of Sivakoff & Saslaw (2005) taken from the 2MASS data. This is important, since the observations have a slightly higher peak than the theoretical distribution, and we want to know if this was caused by neglecting the irreducible triplet term. Figure 6 shows that the distribution functions of equation (58) agree very closely with the observations and with the previous $f_V(N)$ without the irreducible triplets, particularly for larger values of N and $\bar{N}$. After examining a wide range of values of $\bar{N}$, N, and b for a number of different cases, we conclude that the distribution function is not significantly affected by the inclusion of triplet terms of the form (△), which confirms the self-consistency of the earlier theory (Paper I). Therefore, the slightly higher observed peaks have a different origin.

6. DISCUSSION

We have rederived the statistical mechanics and thermodynamics of the gravitational cosmological many-body problem more accurately by including terms for irreducible triplets in the partition function. These represent the close interactions of three galaxies through their pairwise gravitational forces. They have the form $f_{12} f_{13} f_{23}$, indicated by the cluster diagrams (△). These irreducible triplets also contribute the term ξ_{123} to the three-point correlation function. This ξ_{123} term is normally neglected in the BBGKY hierarchy of correlation functions, because its nonlinear calculation is usually mathematically very difficult. Although the irreducible triplets are not trivial to calculate for the partition function, we have done this and shown that their effects are indeed small.

These terms might have turned out to be important, as in some cases of imperfect gases (e.g., Graben & Present 1962). Their comparative unimportance here results partly from the relatively small number of permanent close triple groups in systems with large N. This, in turn, is partly a consequence of the dynamical instability of the gravitational three-body problem. To estimate this, consider $f_3(V)$ from equation (58), which we may write, without the triplet term for simplicity (i.e., for $a_N = 0$), as

$$f_3(x) = \frac{1}{6}x(x+3b)^2 e^{-x-3b}, \tag{59}$$

where

$$x \equiv \frac{4}{3}\pi(1-b)\bar{n}r^3. \tag{60}$$

To find the radius $r_{3\max}$ of the most probable volume in which a group of three galaxies is found, we solve $d \ln f_3(x)/dx = 0$ to obtain

$$r_{3\max}^3 = \frac{9}{8\pi}\left\{1 + \left[1 + \frac{4b}{3(1-b)^2}\right]\right\}^{1/2} \bar{n}^{-1}. \tag{61}$$

This volume exceeds the most probable volume for finding two galaxies by the ratio

$$\frac{r_{3\max}^3}{r_{2\max}^3} = \frac{3}{2}\frac{\left\{1 + \left[1 + 4b/3(1-b)^2\right]\right\}^{1/2}}{\left\{1 + \left[1 + 2b/(1-b)^2\right]\right\}^{1/2}}. \tag{62}$$

This ratio ranges between 3/2 for small values of b and $(3/2)^{1/2}$ as $b \to 1$. Therefore, there will, on average, be fewer volumes containing three galaxies in a typical given region than there are volumes containing two galaxies. More generally, the ratio of the amplitudes $f_3(V)/f_2(V)$ in any volume V is

$$\frac{f_3(V)}{f_2(V)} = \frac{1}{3}[\bar{n}V(1-b) + 3b)]e^{-b}. \tag{63}$$

As $b \to 1$, any volume with $\bar{N} < (1-b)^{-1}$ galaxies is less likely to contain three than two galaxies. Volumes with $\bar{N}(1-b) \geq 1$ will more often contain three galaxies, but the number of such volumes will be fewer, proportional to $(1-b)$, and within them the average separation of the three galaxies will be greater, leading to a smaller contribution of close triplets to the partition function. Moreover, the orbits of these close irreducible triplets are dynamically unstable (e.g., Saslaw et al. 1974), both intrinsically and to external perturbations on timescales typically between 1 and 100 crossing times of their average orbital separation. For irreducible triplets, the orbital separations and timescales are generally smaller than for other triplets. All these effects reduce the contribution of irreducible triplets to the partition function.

Another reason why irreducible triplet configurations are less important for this cosmological many-body system than for some imperfect gases or condensed matter systems is that the fundamental gravitational interaction is always pairwise, rather than directly dependent on the positions of three or more particles, as in other cases. This considerably simplifies the configuration integrals.

A result of this simplification is that some of the thermodynamic properties such as internal energy and pressure for the cosmological case retain the same functional form as they have without irreducible triplets. Only their value of b is modified slightly, and this modification becomes small in the three limits $b \to 0$, $b \to 1$, and $N \to \infty$, as seen in equation (42). The small contribution of this modification may also be seen in Figure 2.

For the specific heat and the $f(N)$ distribution functions, however, the modifications can be greater in some regimes. Including irreducible triplets lowers C_V for $3 \lesssim N \lesssim 20$, particularly if $0.33 \lesssim b \lesssim 0.6$, since those systems typically contain a large relative number of three-particle groups. The irreducible triplets lower C_V, since their internal gravitational energies contribute negatively to it, and for $N \lesssim 5$ they slightly reduce the value of b at which C_V becomes negative, thus promoting instabilities. For the distribution function, equation (58) shows that its form is modified slightly by replacing b in the exponential by b_t and adding extra terms for the normalization. Fits to the 2MASS galaxy catalog are within the noise, as are the fits without the irreducible triplet terms, so they do not significantly alter the close agreement with observations of galaxy clustering. We have also shown how the coefficient of thermal expansion, the isothermal compressibility, and the specific heat at constant pressure are modified by the irreducible triplet configurations.

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